# 経済リスクの統計学の新展開2013

## 稀な事象と再起的事象[1]

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(共編)

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[1] この報告書は文部科学省・科学研究費プロジェクト「経済リスクの統計学の新展開：稀な事象と再起的事象」(2013 年度〜) が 2013 年 12 月 26 日に東京大学経済学研究科において開催した研究集会における講演の内容をまとめたものである。

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# 概要

日本では 2011 年 3 月に発生した東日本大震災を一つの契機に「通常の常識では起こりにくいとされる事象」についてのリスク解析や対策の重要性についての認識が高まっている。経済・社会における近年の現象でも 2008 年に起きたリーマンショック・経済危機、2011 年から経験しているヨーロッパ諸国の金融危機なども我々が暮らしている国際的な経済社会においては、従来の議論ではほとんど考慮されていない経済変動の例である。こうした事前には予想が困難で無視されてきた事象、自然災害、経済変動の中でも実際に起きると大きな影響のある不確実な事象を科学的に理解し、有効な対策を考察する研究が必要であり重要である。本研究プロジェクトでは近年の日本など現代の経済・社会の理解にとって重要になっている「きわめて稀に起きる事象」と「しばしば起きる事象」の評価・分析法について研究する予定である。「稀な事象」に関わる経済リスクの分析という課題について理論的・実証的な観点から分析することにより、科学的根拠にもとづいた経済・社会における「経済リスクの分散化」という方策、公共的政策のあり方の提案することが目標である。近年に特に関心が高まっている「従来の常識では希にしか起きない、無視できると見なされる事象」と「ときどき経済・社会では起きると見なされる現象」の科学的解析を柱に、確率論・統計学と経済学・金融（ファイナンス・保険）における既存の理論と現実の乖離、新しい数理的理論の構築と応用、新しい数理的理論を踏まえた「経済リスクの解析と分散化の方策」について研究活動を行う予定である。本研究プロジェクトでは経済リスクを (i) 社会・人口リスク, (ii) 自然災害と極端な事象のリスク, (iii) 経済・金融・保険の対象となるリスク、に関連した３つの領域の経済リスクに分類し、リスクに係わる問題と相互に関わる総合的問題という二つの方向から問題を理論的に解明し、総合的な研究をふまえた経済リスクの科学的制御・管理の方策を提言することを目指す。さらに、経済統計学における研究・研究者と確率論・統計学など数理科学の関係者、さらに金融（ファイナンス）の関係者を交え、現代の社会・経済においては重要ではあるが、既存の研究分野では十分に取り上げられなかった研究課題を研究するとともに、経済リスクの分析と科学的制御・統計的管理法についての共同研究を行う計画である。

今回の研究集会では、経済リスクの統計学を巡るさまざまなトピックについて報告を行う機会であった。このような情報交換が関係者の知的刺激となり、経済リスクの統計学の今後の展開の一助になることを期待する次第である。

# 研究集会・プログラム

＜セッションI：リスク尺度と統計分析＞
Chair：一場知之 (カリフォルニア大学サンタバーバラ校)

9:50〜10:30「リスク尺度と法則不変性」楠岡成雄 (東京大学)
10:35〜11:15「Backtesting distortion risk measure and its backtestability”」Hideatsu Tsukahara(塚原英敦, 成城大学)
11:20〜12:00「Decreasing Trends in Stock-Bond Correlations」Tatsuyoshi Okimoto(沖本竜義, 一橋大学)

(休憩)

＜セッションII：保険市場と統計分析＞
Chair：松井宗也 (南山大学)

14:00〜14:40「公表データにもとづく損保リスクモデル」 田中周二 (日本大学)
14:45〜15:25「On a generalization from ruin to default in Levy insurance risks」Yasutaka Shimizu(清水泰隆, 大阪大学)

＜セッションIII：高頻度金融データと統計分析＞
Chair：一場知之 (カリフォルニア大学サンタバーバラ校)

15:30〜16:10「先物市場の高頻度データ」川崎能典 (統計数理研究所)
16:15〜16:45「高頻度金融データ分析とシグナル・ノイズ」国友直人 (東京大学)

リスク尺度と法則不変性

楠岡成雄
東京大学大学院数理科学研究科

金融リスクの計量化

考え方： Föllmer の考え方が代表的
１期間モデルを基礎にする

良いレビュー

Hans Föllmer, Thomas Knispel

Convex Risk Measures:
Basic Facts, Law-invariance and beyond,
Asymptotics for Large Portfolios

この中では中心極限定理との関係（アクチュアリー的な問題）
も述べられている

**Backtesting Distortion Risk Measures**

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## Contents

1. Introduction to Distortion Risk Measures (DRMs)

2. Statistical Estimation
   - Asymptotic results
   - Estimation of asymptotic variance
   - Bias correction

3. Backtesting DRMs
   - Unconditional & Conditional approaches
   - Backtestability & Elicitability

## 1. Distrotion Risk Measures

A random variable $X$ represents a loss of some financial position

**DRM**

Any coherent risk measure satisfying law invariance and comonotonic additivity is a **distortion risk measure**:

$$\rho(X) = \rho(F) := \int_{[0,1]} F^{-1}(u)\,\mathrm{d}D(u) = \int_{\mathbb{R}} x\,\mathrm{d}D \circ F(x).$$

where $F$ is the df of $X$, $F^{-1}$ is the quantile of $X$, and $D$ is a convex distortion, i.e., a df on $[0,1]$.

►► a.k.a. spectral risk measure (Acerbi), weighted V@R (Cherny)

**Example**: *Expected Shortfall (ES)*

The expected loss that is incurred when VaR is exceeded:

$$\mathsf{ES}_\theta(X) := \frac{1}{\theta}\int_{1-\theta}^{1} F^{-1}(u)\,\mathrm{d}u \doteq \mathrm{E}(X \,|\, X \geq \mathsf{VaR}_\theta(X))$$

Taking distortion of the form

$$D_\theta^{\mathsf{ES}}(u) = \frac{1}{\theta}\big[u - (1-\theta)\big]_+, \quad 0 < \theta < 1$$

yields ES as a distortion risk measure.

►► Typical values for $\theta$ are: 0.05, 0.01, ...

## Other Examples of DRM:

- *Proportional Hazards*:

$$D_\theta^{\mathsf{PH}}(u) = 1 - (1-u)^\theta,$$

- *Proportional Odds*:

$$D_\theta^{\mathsf{PO}}(u) = \frac{\theta u}{1-(1-\theta)u}$$

- *Gaussian* (Wang transform):

$$D_\theta^{\mathsf{GA}}(u) = \Phi(\Phi^{-1}(u) + \log\theta)$$

★ See Tsukahara (2009) *Mathematical Finance*, vol. 19.

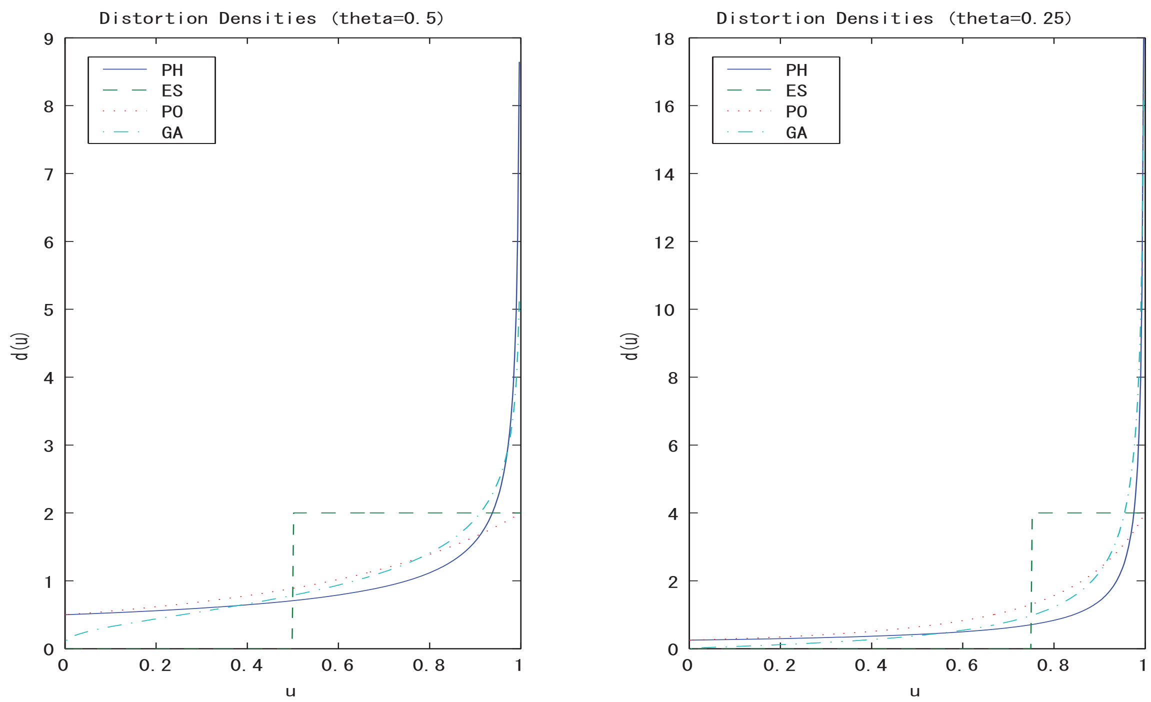


Figure 1: Distortion densities ($\theta = 0.5$, $\theta = 0.25$)

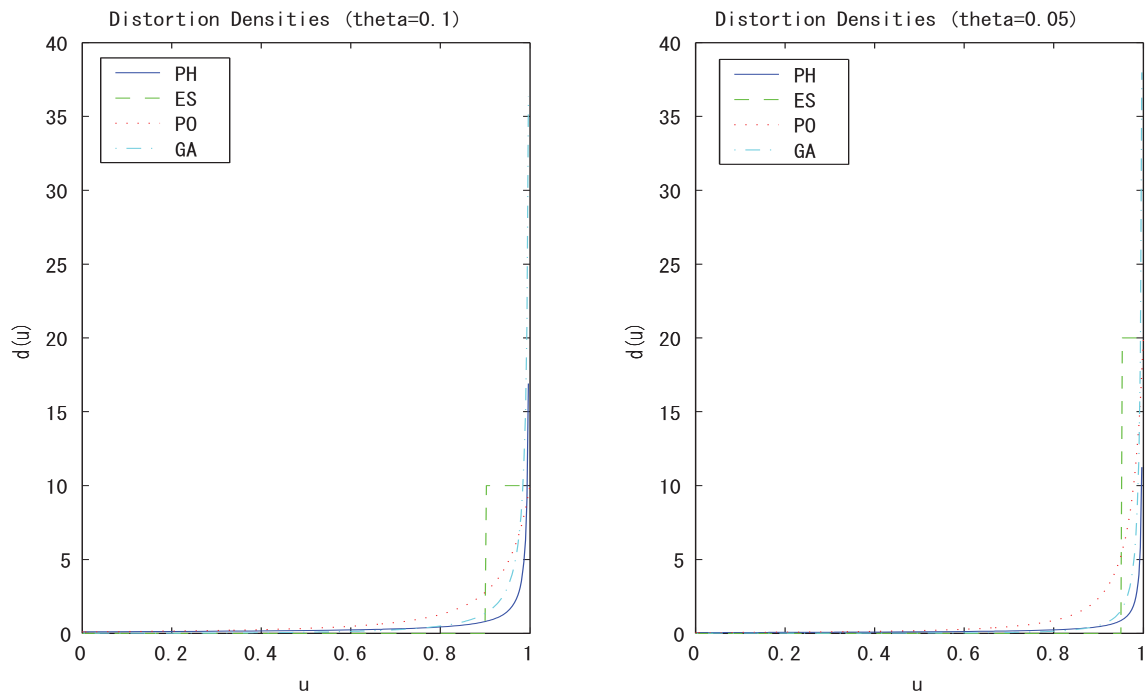


Figure 2: Distortion densities ($\theta = 0.1$, $\theta = 0.05$)

## 2. Statistical Estimation

$(X_n)_{n\in\mathbb{N}}$: strictly stationary process with $X_n \sim F$

$\mathbb{F}_n$: empirical df based on the sample $X_1, \ldots, X_n$

A natural estimator of $\rho(F)$ is

$$\widehat{\rho}_n = \int_0^1 \mathbb{F}_n^{-1}(u)\, dD(u)$$
$$= \sum_{i=1}^{n} c_{ni} X_{n:i}, \qquad c_{ni} := D\left(\frac{i-1}{n}, \frac{i}{n}\right]$$

This type of statistics is called L-statistics

**Strong consistency**

Let $d(u) = \frac{\mathrm{d}}{\mathrm{d}u} D(u)$ for a convex distortion $D$, and $1 \le p \le \infty$, $1/p + 1/q = 1$. Suppose

- $(X_n)_{n \in \mathbb{N}}$ is an ergodic stationary sequence
- $d \in L^p(0,1)$ and $F^{-1} \in L^q(0,1)$

Then

$$\widehat{\rho}_n \longrightarrow \rho(F), \quad \text{a.s.}$$

For a proof, see van Zwet (1980, AP)

[All we need is SLLN and Glivenko-Cantelli Theorem].

<ins>**Assumptions for asymptotic normality**</ins>:

- $(X_n)_{n \in \mathbb{N}}$ is strongly mixing with rate

$$\alpha(n) = O(n^{-\theta-\eta}) \quad \text{for some } \theta \ge 1 + \sqrt{2},\ \eta > 0$$

- For $F^{-1}$-almost all $u$, $d$ is continuous at $u$

- $|d| \le B, \quad B(u) := M u^{-b_1} (1-u)^{-b_2}$,

- $|F^{-1}| \le H, \quad H(u) := M u^{-d_1} (1-u)^{-d_2}$

Assume $b_i$, $d_i$ & $\theta$ satisfy $b_i + d_i + \dfrac{2b_i + 1}{2\theta} < \dfrac{1}{2}, \quad i = 1, 2$

Set

$$\sigma(u,v) := [u \wedge v - uv] + \sum_{j=1}^{\infty}[C_j(u,v) - uv] + \sum_{j=1}^{\infty}[C_j(v,u) - uv],$$

$$C_j(u,v) := \mathrm{P}(X_1 \leq F^{-1}(u), X_{j+1} \leq F^{-1}(v))$$

**Theorem (Asymptotic Normality)**

Under the above assumptions, we have

$$\sqrt{n}(\widehat{\rho}_n - \rho(F)) \ \xrightarrow{\mathcal{L}}\ N(0, \sigma^2),$$

where

$$\sigma^2 := \int_0^1 \int_0^1 \sigma(u,v) d(u) d(v)\, \mathrm{d}F^{-1}(u) \mathrm{d}F^{-1}(v) < \infty$$

- GARCH model:

  $X_t = \sigma_t Z_t, \quad (Z_t):$ i.i.d.

  $\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$

  ►► If the stationary distribution has a positive density around 0, then GARCH is strongly mixing with exponentially decaying $\alpha(n)$

- Stochastic Volatility model:

  $X_t = \sigma_t Z_t, \quad (Z_t):$ i.i.d., $\quad (\sigma_t):$ strictly stationary positive

  $(Z_t)$ and $(\sigma_t)$ are assumed to be independent

  ►► The mixing rate of $(X_t)$ is the same as that of $(\sigma_t)$

## Estimation of Asymptotic Variance

Let

$$Y_n := \int [\mathbf{1}\{X_n \le x\} - F(x)] d(F(x))\,\mathrm{d}x, \quad n \in \mathbb{Z}.$$

Then $Y_n$ is also a strictly stationary and strongly mixing sequence with the same mixing coefficient as $X_n$. Furthermore

$$\mathrm{E}(Y_n) = 0, \qquad \sigma^2 = \sum_{h=-\infty}^{\infty} \gamma(h) < \infty,$$

where $\gamma(h) := \mathrm{E}(Y_n Y_{n+h})$.

Let $f$ be the spectral density of $(Y_n)$. Then

$$\sum_{h=-\infty}^{\infty} \gamma(h) = 2\pi f(0)$$

$\Longrightarrow$ Use a consistent estimator of $f(0)$ (JHB approach)

The lag window estimator is defined by

$$\widehat{f}_n(\lambda) = \frac{1}{2\pi} \sum_{|k|<K_n} w(k/K_n) \widehat{\gamma}_n(k) \cos k\lambda$$

where $w$ is a "lag window", and $\widehat{\gamma}_n(k) := \frac{1}{n} \sum_{i=1}^{n-k} Y_i Y_{i+k}$

►► $F$ in the expression of $Y_n$ is unknown, so we replace it with the empirical df. That is, we use

$$Y_{i,n} := \int [\mathbf{1}\{X_i \le x\} - \mathbb{F}_n(x)] d(\mathbb{F}_n(x))\, \mathrm{d}x, \quad i = 1, \dots, n$$

Let

$$\widetilde{\gamma}_n(k) := \frac{1}{n} \sum_{i=1}^{n-k} Y_{i,n} Y_{i+k,n} \quad \text{and} \quad \widetilde{f}_n(0) := \frac{1}{2\pi} \sum_{|k|<K_n} w(k/K_n) \widetilde{\gamma}_n(k)$$

Then $2\pi \widetilde{f}_n(0)$ should give a consistent estimator of the asymptotic variance $\sigma^2$

**Theorem**

In addition to the conditions assumed in the above theorem, suppose that $J$ is Lipschitz, $w$ is a bounded even function which is continuous in $[-1,1]$ with $w(0) = 1$ and equals 0 outside $[-1,1]$. Also assume $\mathrm{E}|Y_n|^4 < \infty$ and the fourth-order cumulants

$$\kappa(h,i,j) := \mathrm{E}(Y_1 Y_{1+h} Y_{1+i} Y_{1+j}) - \gamma(h)\gamma(i-j) - \gamma(i)\gamma(h-j) - \gamma(j)\gamma(h-i)$$

are summable: $\sum_{h,i,j=-\infty}^{\infty} |\kappa(h,i,j)| < \infty$.

Let $K_n$ be a sequence of integers such that $K_n \to \infty$ and $K_n/\sqrt{n} \to 0$ as $n \to \infty$. Then we have

$$2\pi \widetilde{f}_n(0) \xrightarrow{L_1} \sigma^2, \quad n \to \infty$$

## Bias of $L$-statistics

By Fubini, for any df $F$ and any distortion $D$,

$$\int_{[0,1]} F^{-1}(u)\,\mathrm{d}D(u) = -\int_{\infty}^{0} D(F(x))\,\mathrm{d}x + \int_0^{\infty} [1 - D(F(x))]\,\mathrm{d}x$$

By Fubini and Jensen, for convex $D$,

$$\begin{aligned}
&\mathrm{E}\left[\int_{[0,1]} \mathbb{F}_n^{-1}(u)\,\mathrm{d}D(u)\right] \\
&= \int_{\infty}^{0} \mathrm{E}(-D(\mathbb{F}_n(x)))\,\mathrm{d}x + \int_0^{\infty} \mathrm{E}[1 - D(\mathbb{F}_n(x))]\,\mathrm{d}x \\
&\le \int_{\infty}^{0} -D(\mathrm{E}(\mathbb{F}_n(x)))\,\mathrm{d}x + \int_0^{\infty} [1 - D(\mathrm{E}(\mathbb{F}_n(x)))]\,\mathrm{d}x \\
&= \int_{[0,1]} F^{-1}(u)\,\mathrm{d}D(u)
\end{aligned}$$

Therefore

$$\mathrm{E}(\widehat{\rho}_n) - \rho(F) \le 0$$

$\Longrightarrow$ $\widehat{\rho}_n$ has a negative bias

Need bias correction methods. For the i.i.d. case,

- Xiang (1995): Modify the form of $L$-statistics
- Kim (2010): Bootstrap-based method

►► The bootstrap methodology is still available in the dependent case (see Lahiri (2003), Example 4.8).

## Moving Block Bootstrap (MBB)

- Data: $X_1, \ldots, X_n$
- Block size: $\ell$, # of blocks: $N := n - \ell + 1$
- Blocks: $\mathcal{B}_i = (X_i, \ldots, X_{i+\ell-1})$, $i = 1, \ldots, N$

Resample $k = [n/\ell]$ blocks from $\{\mathcal{B}_1, \ldots, \mathcal{B}_N\}$ with replacement to get $\mathcal{B}_1^*, \ldots, \mathcal{B}_k^*$

Write $\mathcal{B}_i^* = (X^*_{(i-1)\ell+1}, \ldots, X^*_{i\ell})$

$\Longrightarrow X_1^*, \ldots, X_{k\ell}^*$: MBB sample

MBB version of $\widehat{\rho}_n$ is

$$\widehat{\rho}_n^* = \frac{1}{n} \sum_{i=1}^{n} c_{ni} X^*_{n:i}, \qquad c_{ni} := D\left(\frac{i-1}{n}, \frac{i}{n}\right]$$

Validity of MBB follows from an argument specific to our case.

►► The approach based on Hadamard differentiability of $L$-functional

$$T(F) := \int_0^1 h(F^{-1}(u)) J(u) \, \mathrm{d}u$$

is not convenient. See Boos (1979, AS), Lahiri (2003), Section 12.3.5.

## Simulation example: inverse-gamma SV model

$$X_t = \sigma_t Z_t$$

$Z_t$ i.i.d. N(0,1) and $V_t = 1/\sigma_t^2$ satisfies

$$V_t = \rho V_{t-1} + \varepsilon_t,$$

where $V_t \sim$ Gamma$(a, b)$ for each $t$, $(\varepsilon_t)$ i.i.d. rv's, and $0 \le \rho < 1$

$\Rightarrow X_t$ has scaled $t$-distribution with $\nu = 2a$, $\sigma^2 = b/a$

►► Lawrance (1982): the distribution of $\varepsilon_t$ is compound Poisson

►► Can be shown that $(X_t)$ is geometrically ergodic

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Simulation results for estimating VaR, ES & PO risk measures with inverse-gamma SV observations ($n = 500$, # of replications $= 1000$)

$X_t = \sigma_t Z_t$, where $V_t = 1/\sigma_t^2$ follows AR(1)
with gamma(2,16000) marginal & $\rho = 0.5$, $Z_t$ i.i.d. N(0,1)

| | | VaR | | ES | | PO | |
|---|---|---|---|---|---|---|---|
| | $\theta$ | bias | RMSE | bias | RMSE | bias | RMSE |
| | 0.1 | 0.0692 | 10.9303 | −2.2629 | 22.1361 | −1.7739 | 17.5522 |
| SV | 0.05 | 2.5666 | 17.6755 | −1.2168 | 37.2719 | −2.0200 | 28.5053 |
| | 0.01 | 14.9577 | 61.2290 | −11.9600 | 103.9269 | −15.7888 | 73.7147 |
| | 0.1 | 0.7976 | 10.5893 | −1.2914 | 19.5756 | −1.3574 | 15.3271 |
| i.i.d. | 0.05 | 0.7974 | 16.1815 | −2.6346 | 31.3166 | −2.8342 | 23.9933 |
| | 0.01 | 10.6838 | 53.2567 | −12.9355 | 95.9070 | −15.8086 | 69.5425 |

## Simulation results for estimating variance and bias of PO risk measure

($n = 500$, $K_n = 5$, Parzen kernel $w(x) = 1 - x^2$, block size$= 5$, # of bootstrap replicates $= 800$, # of replications $= 10000$)

| | $\rho$ | $\theta$ | MC bias | MC s.e. | $\widehat{\text{A-s.e.}}$ | BS bias | BS s.e. |
|---|---|---|---|---|---|---|---|
| IG-SV | | 0.1 | −0.8328 | 15.4456 | 14.0956 | −0.8151 | 13.9829 |
| $\alpha = 2$ | 0.1 | 0.05 | −2.0580 | 24.6961 | 20.9719 | −1.8170 | 20.6863 |
| $\beta = 16000$ | | 0.01 | −13.3608 | 68.9197 | 46.6943 | −10.2030 | 46.0788 |
| IG-SV | | 0.1 | −0.3345 | 10.7979 | 10.4231 | −0.6812 | 10.3933 |
| $\alpha = 4$ | 0.1 | 0.05 | −1.3663 | 15.1946 | 14.0623 | −1.3511 | 13.9725 |
| $\beta = 48000$ | | 0.01 | −6.8659 | 34.4725 | 26.4183 | −6.0749 | 26.4446 |
| IG-SV | | 0.1 | −0.5432 | 9.0853 | 8.8370 | −0.6048 | 8.8281 |
| $\alpha = 10$ | 0.1 | 0.05 | −1.1786 | 11.7923 | 11.2289 | −1.1263 | 11.2003 |
| $\beta = 144000$ | | 0.01 | −5.8673 | 22.9686 | 18.7767 | −4.4474 | 18.9614 |

| | $\rho$ | $\theta$ | MC bias | MC s.e. | $\widehat{\text{A-s.e.}}$ | BS bias | BS s.e. |
|---|---|---|---|---|---|---|---|
| IG-SV | | 0.1 | −1.0054 | 17.5469 | 15.0711 | −0.8793 | 14.6925 |
| $\alpha = 2$ | 0.5 | 0.05 | −2.2714 | 27.1465 | 22.0852 | −1.9450 | 21.4374 |
| $\beta = 16000$ | | 0.01 | −13.9208 | 74.8887 | 47.7943 | −10.6541 | 46.8379 |
| IG-SV | | 0.1 | −0.5791 | 11.4856 | 10.7162 | −0.6957 | 10.5906 |
| $\alpha = 4$ | 0.5 | 0.05 | −1.3472 | 15.7116 | 14.4718 | −1.3994 | 14.2658 |
| $\beta = 48000$ | | 0.01 | −7.4680 | 35.1014 | 26.7575 | −6.1939 | 26.7115 |
| IG-SV | | 0.1 | −0.8213 | 9.2632 | 8.9299 | −0.6062 | 8.8957 |
| $\alpha = 10$ | 0.5 | 0.05 | −1.0663 | 11.9443 | 11.3608 | −1.1368 | 11.2996 |
| $\beta = 144000$ | | 0.01 | −5.7987 | 23.1130 | 18.8147 | −4.4769 | 18.9896 |

| | $\rho$ | $\theta$ | MC bias | MC s.e. | $\widehat{\text{A-s.e.}}$ | BS bias | BS s.e. |
|---|---|---|---|---|---|---|---|
| IG-SV | | 0.1 | −2.0408 | 28.2224 | 15.5015 | −0.9609 | 14.7212 |
| $\alpha = 2$ | 0.9 | 0.05 | −4.8204 | 42.1005 | 22.1388 | −2.0483 | 20.9685 |
| $\beta = 16000$ | | 0.01 | −23.5844 | 106.4374 | 43.6402 | −10.1556 | 42.4681 |
| IG-SV | | 0.1 | −1.1973 | 14.9586 | 11.1112 | −0.7274 | 10.8092 |
| $\alpha = 4$ | 0.9 | 0.05 | −2.2346 | 20.8199 | 14.8937 | −1.4366 | 14.4566 |
| $\beta = 48000$ | | 0.01 | −10.2968 | 42.5085 | 26.3137 | −6.1439 | 26.0855 |
| IG-SV | | 0.1 | −0.5956 | 10.3666 | 9.1248 | −0.6262 | 9.0293 |
| $\alpha = 10$ | 0.9 | 0.05 | −1.4212 | 13.6534 | 11.5934 | −1.1609 | 11.4494 |
| $\beta = 144000$ | | 0.01 | −6.3827 | 25.2688 | 18.8986 | −4.4824 | 19.0079 |

| | $\rho$ | $\theta$ | MC bias | MC s.e. | $\widehat{\text{A-s.e.}}$ | BS bias | BS s.e. |
|---|---|---|---|---|---|---|---|
| | | 0.1 | −0.5734 | 8.2886 | 8.0638 | −0.5619 | 8.0667 |
| $N(0, 126.5^2)$ | iid | 0.05 | −1.1557 | 10.1327 | 9.8175 | −1.0116 | 9.8117 |
| | | 0.01 | −4.4730 | 18.1714 | 14.9659 | −3.6136 | 15.2192 |
| | | 0.1 | −0.9038 | 15.3536 | 13.9544 | −0.8121 | 13.8815 |
| $t_4(0, 126.5^2)$ | iid | 0.05 | −1.8468 | 24.3247 | 20.8781 | −1.7928 | 20.6468 |
| | | 0.01 | −12.5608 | 73.3170 | 46.9313 | −10.2243 | 46.3147 |
| | | 0.1 | −0.5538 | 10.7575 | 10.3154 | −0.6687 | 10.2909 |
| $t_8(0, 126.5^2)$ | iid | 0.05 | −1.4518 | 14.9271 | 13.9883 | −1.3379 | 13.9033 |
| | | 0.01 | −6.8385 | 34.8496 | 26.4076 | −6.8385 | 26.4531 |
| | | 0.1 | −0.5470 | 9.0123 | 8.8209 | −0.5985 | 8.8127 |
| $t_{20}(0, 126.5^2)$ | iid | 0.05 | −1.1266 | 11.6915 | 11.2178 | −1.1176 | 11.1965 |
| | | 0.01 | −5.5631 | 22.9298 | 18.7808 | −4.4588 | 18.9697 |

## 3. Backtesting

Purpose of Backtesting:

1. Monitor the performance of the model and estimation methods for risk measurement
2. Compare relative performance of the models and methods

**Idea**

ex ante risk measure forecasts from the model

vs.

ex post realized portfolio loss

### Setup

Entire observations: $X_1, \dots, X_T$

Estimation window size $= n$, $\quad m := T - n$

| | data | estimand | realized loss |
|---|---|---|---|
| 1. | $X_1, \dots, X_n$ | $\rho(X_{n+1})$ | $X_{n+1}$ |
| 2. | $X_2, \dots, X_{n+1}$ | $\rho(X_{n+2})$ | $X_{n+2}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $m$. | $X_{T-n}, \dots, X_{T-1}$ | $\rho(X_T)$ | $X_T$ |

## Two approaches to risk measurement

Assume that the loss process $(X_t)_{t\in\mathbb{Z}}$ is a stationary time series with stationary df $F$. At time $t$, we have two options:

### I. Unconditional Approach

Look at the risk measure associated with $F(x) = \mathrm{P}(X_{t+1} \le x)$
(For a large time horizon; credit risk and insurance)

### II. Conditional Approach

For a given filtration $\mathscr{F}_t$, look at the risk measure associated with the conditional df $F_t(x) := \mathrm{P}(X_{t+1} \le x \,|\, \mathscr{F}_t)$,
(For a short time horizon; market risk)

## In the case of VaR

- Unconditional VaR, denoted by $\mathrm{VaR}_\alpha$, satisfies

$$\mathrm{E}\big(\mathbf{1}\{X_{t+1} \ge \mathrm{VaR}_\alpha\}\big) = \alpha$$

  But $\mathbf{1}\{X_{t+1} \ge \mathrm{VaR}_\alpha\}$'s might not be independent

- Conditional VaR, denoted by $\mathrm{VaR}_\alpha^t$, satisfies

$$\mathrm{E}\big(\mathbf{1}\{X_{t+1} \ge \mathrm{VaR}_\alpha^t\} \,|\, \mathscr{F}_t\big) = \alpha$$

By Lemma 4.29 of MFE, if $(Y_t)$ is a sequence of Bernoulli rv's adapted to $(\mathscr{F}_t)$ and if $\mathrm{E}(Y_{t+1} \,|\, \mathscr{F}_t) = p > 0$, then $(Y_t)$ must be i.i.d.

Therefore $\mathbf{1}\{X_{t+1} \geq \mathrm{VaR}_\alpha^t\}$, $t = n, \ldots, T-1$ are i.i.d. Bernoulli rv's.

$$\Downarrow$$

This gives the grounds for backtesting using $\mathbf{1}\{X_{t+1} \geq \widehat{\mathrm{VaR}}_\alpha^t\}$, where $\widehat{\mathrm{VaR}}_\alpha^t$ is an estimate of the VaR associated with the conditional df $F_t(x) := \mathrm{P}(X_{t+1} \leq x \,|\, \mathscr{F}_t)$. Namely,

(i) Test $\displaystyle\sum_{t=n}^{T-1} \mathbf{1}\{X_{t+1} \geq \widehat{\mathrm{VaR}}_\alpha^t\} \sim \mathrm{Bin}(m, \alpha)$

(ii) Test independence of $\mathbf{1}\{X_{t+1} \geq \widehat{\mathrm{VaR}}_\alpha^t\}$, $t = n, \ldots, T-1$ (e.g., runs test)

## Backtesting DRMs

Note that, with $d(u) = \frac{\mathrm{d}}{\mathrm{d}u} D(u)$ and $X \sim F$,

$$\rho(X) = \int_{-\infty}^{\infty} x \,\mathrm{d}D \circ F(x) = \int_{-\infty}^{\infty} x d(F(x)) \,\mathrm{d}F(x)$$

$$= \mathrm{E}[X d(F(X))]$$

Thus $X d(F(X)) - \rho(X)$ has mean 0 unconditionally.

►► In the conditional case, $\mathrm{E}[X_{t+1} d(F_t(X_{t+1})) \,|\, \mathscr{F}_t] = \rho_t(X_{t+1})$, but this does not help much.

## I.I.D. case (rough-and-ready)

If $X_1, \dots, X_T$ are i.i.d. with df $F$, then we can base the backtesting of our method/model on

$$X_{n+1} d(\widehat{\mathbb{F}}_{1:n}(X_{n+1})) - \widehat{\rho}_{(1:n)},$$

$$\vdots$$

$$X_T d(\widehat{\mathbb{F}}_{T-n:T-1}(X_T)) - \widehat{\rho}_{(T-n:T-1)}$$

where $\widehat{\mathbb{F}}_{k:l}$ and $\widehat{\rho}_{(k:l)}$ are estimates based on the sample $X_k, \dots, X_l$

►► If we have dependent data or we use the conditional approach, it is necessary to introduce more explicit time series models.

## Conditional Approach

Write $\rho_t(X_{t+1})$ for a distortion risk measure with a distortion $D$ for the conditional df $F_t(x) := \mathrm{P}(X_{t+1} \le x \mid \mathscr{F}_t)$, $\mathscr{F}_t := \sigma(X_s \colon s \le t)$:

$$\rho_t(X_{t+1}) := \int_{[0,1]} F_t^{-1}(u)\, \mathrm{d}D(u)$$

**Assumption**

Suppose that for $\mathscr{F}_{t-1}$-measurable $\mu_t$ and $\sigma_t$,

$$X_t = \mu_t + \sigma_t Z_t,$$

where $(Z_t)$ is i.i.d. with finite 2nd moment.

## Example: ARMA($p_1$, $q_1$) with GARCH($p_2$, $q_2$) errors

Let $(Z_t)$ be i.i.d. with finite 2nd moment.

$$
\begin{aligned}
X_t &= \mu_t + \sigma_t Z_t, \\
\mu_t &= \mu + \sum_{i=1}^{p_1} \phi_i (X_{t-i} - \mu) + \sum_{j=1}^{q_1} \theta_j (X_{t-j} - \mu_{t-j}), \\
\sigma_t^2 &= \alpha_0 + \sum_{i=1}^{p_2} \alpha_i (X_{t-i} - \mu_{t-i})^2 + \sum_{j=1}^{q_2} \beta_j \sigma_{t-j}^2,
\end{aligned}
$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$, $i = 1, \ldots, p_2$, $\beta_j \geq 0$, $j = 1, \ldots, q_2$.

Usually, it is assumed that $(X_t)$ is covariance stationary,
and $\sum_{i=1}^{p_2} \alpha_i + \sum_{j=1}^{q_2} \beta_j < 1$.

By (conditional) translation equivariance and positive homogeneity,

$$\rho_t(X_{t+1}) = \mu_{t+1} + \sigma_{t+1}\rho(Z)$$

where $Z$ is a generic rv with the same df $G$ as $Z_t$'s.

(i) If $G$ is a known df, $\rho(Z)$ is a known number.
We need to estimate $\mu_{t+1}$ and $\sigma_{t+1}$ based on $X_{t-n+1}, \ldots, X_t$ using some specific model and method (e.g., ARMA with GARCH errors using QML). Then the risk measure estimate is given by

$$\widehat{\rho}_t(X_{t+1}) := \widehat{\mu}_{t+1} + \widehat{\sigma}_{t+1}\rho(Z)$$

Observe that

$$\rho(Z) = \mathrm{E}\left[Z_{t+1} d(G(Z_{t+1}))\right]$$

$$\Downarrow$$

$$\mathrm{E}\left[(Z_{t+1} - \rho(Z)) d(G(Z_{t+1}))\right] = 0$$

Defining

$$R_{t+1} := Z_{t+1} - \rho(Z) = \frac{X_{t+1} - \rho_t(X_{t+1})}{\sigma_{t+1}}$$

one sees that $(R_t d(G(Z_t)))_{t\in\mathbb{Z}}$ is i.i.d.

This suggests that in practice, we may perform backtesting by examining mean-zero behavior of $\widehat{R}_{t+1} d(G(\widehat{Z}_{t+1}))$, $t = n, \ldots, T-1$, where

$$\widehat{R}_{t+1} := \frac{X_{t+1} - \widehat{\rho}_t(X_{t+1})}{\widehat{\sigma}_{t+1}}$$

and

$$\widehat{Z}_{t+1} = \frac{X_{t+1} - \widehat{\mu}_{t+1}}{\widehat{\sigma}_{t+1}} = \widehat{R}_{t+1} + \rho(Z)$$

►► Bootstrap test can be used

(ii) When $G$ is unknown, we need to estimate $G$ in addition to $\mu_{t+1}$ and $\sigma_{t+1}$.

In ARMA with GARCH errors model, we could use the empirical df based on the residuals $\widetilde{Z}_s$'s: for $s = t - n + 1, \dots, t$,

$$\widetilde{Z}_s = \widetilde{\varepsilon}_s / \widetilde{\sigma}_s, \quad \widetilde{\varepsilon}_s : \text{residual from ARMA part}$$

and

$$\widetilde{\sigma}_s^2 = \widehat{\alpha}_0 + \sum_{i=1}^{p_2} \widehat{\alpha}_i \widetilde{\varepsilon}_{s-i}^2 + \sum_{j=1}^{q_2} \widehat{\beta}_j \widehat{\sigma}_{s-j}^2,$$

Then

$$\widetilde{G}_t(z) = \frac{1}{n} \sum_{s=t-n+1}^{t} \mathbf{1}\{\widetilde{Z}_s \le z\},$$

## Simulation study

Simulate GARCH(1,1) process:

$$Y_t = \sigma_t Z_t, \quad Z_t \sim N(0,1) \text{ i.i.d.}$$

$$\sigma_t^2 = 0.01 + 0.9\sigma_{t-1}^2 + 0.08 Y_{t-1}^2$$

Set $T = 1000$, $n = 500$ and $\theta = 0.05$

For $t = n + 1, \dots, T$, plot

(i) $X_t d(\widehat{\mathbb{F}}_{t-n:t-1}(X_t)) - \widehat{\rho}_{(t-n:t-1)}$ (historical, unconditional)

(ii) $\widehat{R}_t d(G(\widehat{Z}_t))$ (normal-GARCH based, conditional)

(i) mean $= -0.0286$, std $= 2.073$

(ii) mean $= -0.0185$, std $= 1.019$

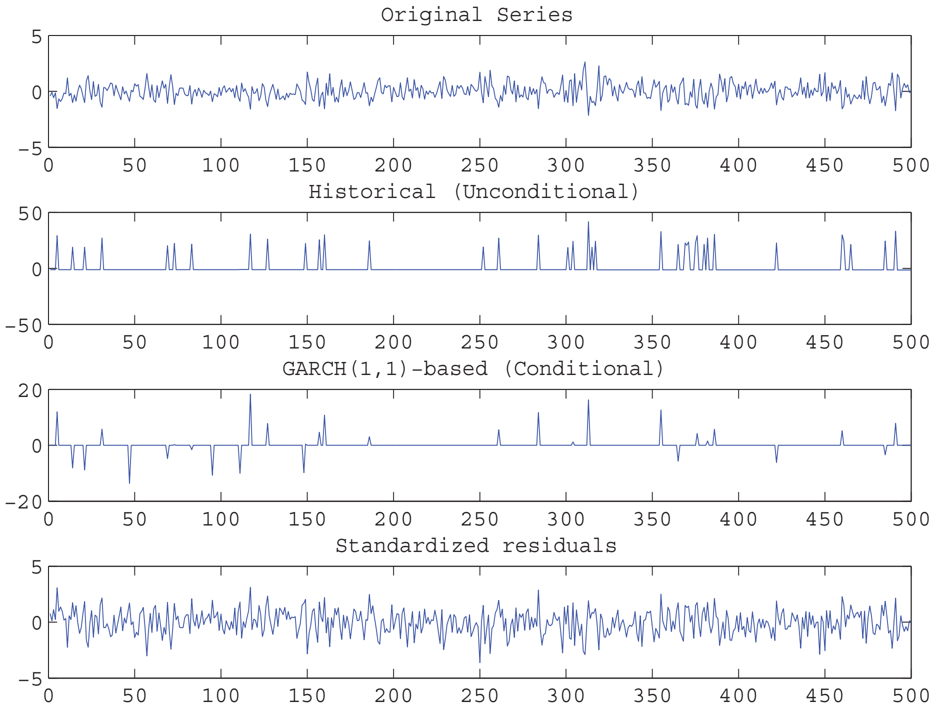


Figure 3: Backtesting results for expected shortfall ($\theta = 0.05$)

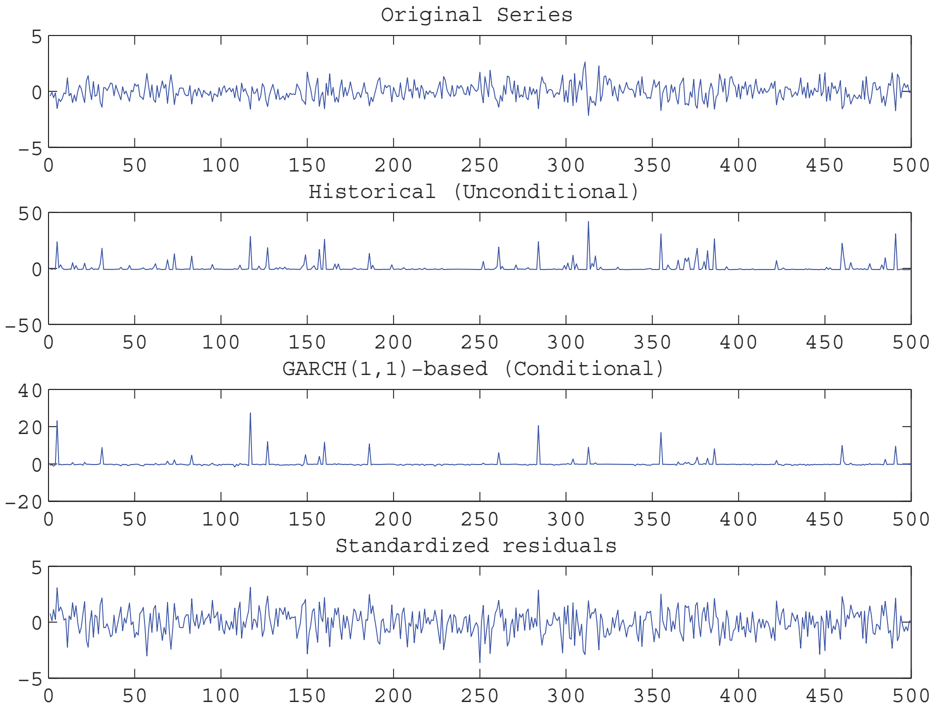


Figure 4: Backtesting results for proportional odds distortion ($\theta = 0.05$)

## Issue: Backtestability

"*It is more difficult to backtest a procedure for calculating expected shortfall than it is to backtest a procedure for calculating VaR*" (Yamai & Yoshiba, Hull, Daníelsson, among others)

1. Because the existing tests for ES are based on
   - parametric assumptions for the null distribution
   - asymptotic approximation for the null distribution
2. Because testing an expectation is harder than testing a single quantile.

## Elicitability

"*Expected shortfall (and spectral risk measures) cannot be backtested because it fails to satisfy* **elicitability** *condition*" (Paul Embrechts, Mar 2013, *Risk Magazine*)

**Def (Osband 1985; Gneiting 2011, JASA)**

A statistical functional $T(F)$ is called **elicitable** r.t. $\mathscr{F}$ if $T(F)$ is a unique minimizer of $t \mapsto E^F[S(t, Y)]$ for some scoring function $S$, $\forall F \in \mathscr{F}$.

## Examples

- $\text{VaR}_\theta(F) = F^{-1}(1-\theta)$ is the unique minimizer for

$$S(t,y) = [\mathbf{1}\{t \le y\} - \theta](y-t)$$
$$= \begin{cases} \theta|y-t| & \text{if } t > y \\ (1-\theta)|y-t| & \text{if } t \le y \end{cases}$$

$\mathscr{F} = \{F\colon \text{absolutely continuous}, \ \int |y| \,\mathrm{d}F(y) < \infty\}$.

- Mean functional $T(F) = \int y \,\mathrm{d}F(y)$ is the unique minimizer for

$$S(t,y) = (y-t)^2$$

$\mathscr{F} = \{F\colon \ \int y^2 \,\mathrm{d}F(y) < \infty\}$.

It is useful when one wants to compare and rank several estimation procedures: With forecasts $x_i$ and realizations $y_i$, use

$$\frac{1}{n}\sum_{i=1}^{n} S(x_i, y_i)$$

as a performance evaluation criterion.

►► But there seems to be no clear connection with backtestability

e.g., mean cannot be backtested nonparametrically based on the sum of squared errors without invoking asymptotic approximation or assuming parametric distribution.

## Basel Committee on Banking Supervision: Consultative Document (October 2013)

> "Move from Value-at-Risk (VaR) to Expected Shortfall (ES):
> A number of weaknesses have been identified with using VaR for determining regulatory capital requirements, including its inability to capture "tail risk". For this reason, the Committee proposed in May 2012 to replace VaR with ES. ES measures the riskiness of a position by considering both the size and the likelihood of losses above a certain confidence level. The Committee has agreed to use a 97.5% ES for the internal models-based approach and has also used that approach to calibrate capital requirements under the revised market risk standardised approach"

## Basel Committee on Banking Supervision: Consultative Document (October 2013)

Backtesting assessment (in *Revised Models-based Approach*):

> "In addition to P&L attribution, the performance of a trading desk's risk management models will be evaluated through daily backtesting. Backtesting requirements would be based on comparing each desk's 1-day static value-at-risk measure at both the 97.5th percentile and the 99th percentile to actual P&L outcomes, using at least one year of current observations of the desk's one-day actual and theoretical P&L. The backtesting assessment would be run at each trading desk as well as for the global (bank-wide) level."

## Concluding Remarks

- Estimation of DRMs is possible with time series data, but for some DRMs, we do not get nice asymptotic properties.

- Backtesting procedure can be performed with DRMs. May need more rigorous/effective procedures.

- Euler capital allocation based on DRMs are easy to compute and widely applicable (with importance sampling)

- Most of the estimation part is published in *Journal of Financial Econometrics* (2013, online)

設定

$\Omega$ ：シナリオの集合（今後考えられる経済変動等のシナリオすべて）

$\mathcal{F}$ : $\Omega$ 上の $\sigma$-加法族　（数学技術の理由から設定）

$X(\omega)$ : シナリオ $\omega$ が起きた時の (事後の) 会社資産価値

$X \in m(\mathcal{F})$ ：$\mathcal{F}$-可測関数　（数学技術の理由からの仮定）

$X(\omega)$ は事前のポートフォリオにより決まる

許容できる事後の資産状況 $\mathcal{A} \subset m(\mathcal{F})$

$\mathcal{A}$ の満たすべき性質

(1) $0 \in \mathcal{A}$

(2) $X \in \mathcal{A},\ X \leqq Y \Rightarrow Y \in \mathcal{A}$

付加的な仮定

(凸性の仮定) $X, Y \in \mathcal{A},\ \lambda \in (0,1) \Rightarrow \lambda X + (1-\lambda)Y \in \mathcal{A}$

(正 1 次同次性) $X \in \mathcal{A},\ \lambda > 0 \Rightarrow \lambda X \in \mathcal{A}$

［計量化］

(1) 対象となる $m(\mathcal{F})$ の部分ベクトル空間 $\mathcal{X}$ を設定

(2) $\rho : \mathcal{X} \to [-\infty, \infty]$ を以下で定義

$$\rho(X) = \inf\{a \in \mathbf{R};\ a + X \in \mathcal{A}\}, \qquad X \in \mathcal{X}$$

（通常は $\rho(\mathcal{X}) \subset \mathbf{R}$ となるように設定される）

$\rho(X)$ はポートフォリオを変えないならば第 0 期に必要な資本と解釈

（金利はここでは無視する）

$\mathcal{A}_\rho = \{X \in \mathcal{X};\ \rho(X) \geqq 0\}$ ：これも許容できるものの集合

$\rho : \mathcal{X} \to \mathbf{R}$ の性質

(0)（ゼロルール）$\rho(0) = 0$

(1) (平行移動不変性）$\rho(X + c) = \rho(X) - c, \quad X \in \mathcal{X},\, c \in \mathbf{R}$

(2)（単調性）$X, Y \in \mathcal{X},\, X \leqq Y \Rightarrow \rho(X) \geqq \rho(Y)$

上記の (0), (1),(2) を満たすものを　リスク尺度と呼ぶ

(凸性の仮定)

$X, Y \in \mathcal{X},\, \lambda \in (0, 1) \Rightarrow \rho(\lambda X + (1 - \lambda)Y) \leqq \lambda\rho(X) + (1 - \lambda)\rho(Y)$

(正一次同次性の仮定)

$X \in \mathcal{X},\, \lambda > 0 \Rightarrow \rho(\lambda X) = \lambda\rho(X)$

凸性、正一次同次性を持つものリスク尺度： coherent risk measure

凸性、正一次同次性の仮定は妥当か？

凸性は自然な面がある（後述）

Föllmer は同次性にはあまり重きを置いていない

正一次同次性は危険という実務家もいる

過去データをどのように考えるべきか！

確率の概念の導入

$P$ : $(\Omega, \mathcal{F})$ 上の確率測度

(法則不変性 (law invariant) の仮定)

$X, Y \in \mathcal{X},\, X, Y$ の分布が $P$ の下で等しい $\Rightarrow \rho(X) = \rho(Y)$

法則不変性は問題を単純化するという利点がある

これまで提案された具体的なリスク尺度は ほぼすべて法則不変性をもつ！

Value at Risk

Average Value at Risk,

(Conditinal Tail Expectation, Expected Shortfall)

shortfall risk measures

divergence risk measures

Haezendonck risk measures

the entropic risk measures

VaR　一次同次性も持つ、　凸性は持たない

AVaR　coherent risk measure

法則不変性、凸性を持つリスク尺度

基本 AVaR

$\mathrm{AVaR}_\lambda$, $\lambda \in [0,1]$

$$\mathrm{AVaR}_\lambda(X) = \frac{1}{\lambda} \inf_{z \in \mathbf{R}} E^P[(z-X)^+ - \lambda z], \qquad \lambda \in (0,1]$$

$$\mathrm{AVaR}_0(X) = ess.sup(-X)$$

分布関数 $F$ に対して

$$R_\lambda(F) = \frac{1}{\lambda} \inf_{z \in \mathbf{R}} \int_{\mathbf{R}} (z-x)^+ dF - \lambda z), \qquad \lambda \in (0,1]$$

$$R_0(F) = \sup\{x \in \mathbf{R};\ F(-x) > 0\}$$

とおけば

$$\mathrm{AVaR}_\lambda(X) = R_\lambda(F_X^P), \qquad \lambda \in [0,1]$$

AVaR は Building Block

$\mathcal{M}_1([0,1])$ $[0,1]$ 上の確率測度全体

**定理 1 (基本定理)** 法則不変性、凸性を持つリスク尺度 $\rho$ に対して $\beta^{\min} : \mathcal{M}_1([0,1]) \to [0,\infty]$ が存在して

$$\rho(X) = \sup_{\mu \in \mathcal{M}_1([0,1])} (\int_{[0,1]} \mathrm{AVaR}_\lambda(X)\mu(d\lambda) - \beta^{\min}(\mu)).$$

$$R_\rho(F) = \sup_{\mu \in \mathcal{M}_1([0,1])} (\int_{[0,1]} R_\lambda(F)\mu(d\lambda) - \beta^{\min}(\mu))$$

とおけば

$$\rho(X) = R_\rho(F_X^P).$$

「リーマンショックに対する反省」

リスク規制が役に立たなかった

100 年に一度の出来事だから仕方ない（100 年に一度とは本当か？）

$P$ が所与（あるいは知ることが出来る）と考えたことが誤りではないか

モデルリスク

(1) 統計推測が誤っているリスク（通常はこれを指す)

(2) モデルが根本的に間違っているというリスク

(1) のタイプのモデルリスク

データをある程度信頼している　（平時を想定してのリスク管理)

(2) のタイプのモデルリスク

データを信頼しない　（金融危機を想定してのリスク管理)

Föllmer の Beyond Law-Invariance

(1) のタイプのモデルリスクを想定していると見える（かつネイマン的）

例えば　$\rho$ が法則不変なリスク尺度とする時、
それは $X$ の $P$ の下での分布 $F_X^P$ の関数となる、則ち

$$\rho(X) = R(F_X^P), \qquad X \in \mathcal{X}$$

であるが $\mathcal{Q}$ を $P$ を含む　$(\Omega, \mathcal{F})$ 上の確率測度よりなる族とした時

$$\tilde{\rho}(X) = \sup\{R(F_X^Q);\ Q \in \mathcal{Q}\}$$

を考えるという発想

$P$ を推定値とすると
$\mathcal{Q}$ : いわば信頼区間のようなもの

ベイズ的やり方も考え得る （一例）

$$\rho^P(X) = R(F_X^P), \qquad X \in \mathcal{X}$$

元々考えている法則不変なリスク尺度

$\mathcal{P}$ を $(\Omega, \mathcal{F})$ 上の確率測度よりなる族とし、
$\nu$ を $\mathcal{P}$ 上の確率測度（事後分布）とした時

$$\tilde{\rho}(X) = \tilde{R}(G)$$

ただし、 $G$ は $-R(F_X^Q)$ $(= -\rho^Q(X))$ の $\nu(dQ)$ の下での分布関数
$\tilde{\rho}(X)$ は凸性を持つリスク尺度
$-\rho^Q(\lambda X + (1-\lambda)Y) \geqq -\lambda\rho^Q(X) - (1-\lambda)\rho^Q(Y)$

(2) のタイプのモデルリスクに対してはどうすればよいか
基本的にはストレスシナリオによるストレステストの考え方
となるのではないか

(1)　ストレスシナリオとは何か？

(2)　合格基準は何か

通常のストレステスト

$\omega_1, \omega_2, \ldots, \omega_n \in \Omega$　ストレスシナリオ

$X \in \mathcal{A}$ の条件： $X(\omega_k) \geqq c_k,\ k = 1, \ldots, n$ (合格基準)

シナリオが限定的

$X(\omega)$ が $\omega$ について「ロバスト」（連続）でないと意味がない

ストレステストの一般化（以下は一例）

基本定理をもう一度見てみる

$Q$： $(\Omega, \mathcal{F})$ 上の確率測度とする

$\mathrm{AVaR}_\lambda^Q,\ \lambda \in [0,1]$ を

$$\mathrm{AVaR}_\lambda^Q = \frac{1}{\lambda} \inf_{z \in \mathbf{R}} E^Q[(z - X)^+ - \lambda z], \qquad \lambda \in (0,1]$$

$$\mathrm{AVaR}_0^Q = Q\text{-}ess.sup(-X)$$

で定めると

$\mathrm{AVaR}_\lambda^Q,\ \lambda \in [0,1]$, は coherent risk measure

$\beta : \mathcal{M}_1([0,1]) \to [0,\infty]$　に対して

$$\rho^{Q,\beta}(X) = \sup_{\mu \in \mathcal{M}_1([0,1])} \left( \int_{[0,1]} \mathrm{AVaR}_\lambda^Q(X) \mu(d\lambda) - \beta(\mu) \right).$$

と定める

$\rho^{Q,\beta}$ は平行移動不変性、単調性、凸性、$Q$-法則不変性　を持ち

$$\rho^{Q,\beta}(0) \leqq 0$$

$Q_k$, $k = 1, \ldots, n$, $(\Omega, \mathcal{F})$ 上の確率測度（リスクシナリオ）
$\beta_k : \mathcal{M}_1([0,1]) \to [0, \infty]$, $k = 1, \ldots, n$, （合格基準）

$$\tilde{\rho}(X) = \max\{\rho^{Q_k,\beta_k}(X) - a_k;\ k = 1, \ldots, n\}$$

とおくと $\tilde{\rho}$ 平行移動不変性、単調性、凸性を持ち
$\tilde{\rho}(0) \leqq 0$
平時の凸性を持つリスク尺度 $\rho_0$
$\rho = \tilde{\rho} \vee \rho_0$ は凸性を持つリスク尺度

課題

統計的推測の誤りを考慮したリスク尺度として何がよいか

ストレステストの一般化として適切なものは何か

まったく異種のリスクが組み合わさった時のリスクの考え方

$(\Omega_i, \mathcal{F}_i, P_i)$, $i = 1, 2$,
$(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, P_1 \otimes P_2)$
$\rho_i$, $i = 1, 2$ $(\Omega_i, \mathcal{F}_i, P_i)$ 上のリスク
「　$\rho_1 \otimes \rho_2$　」をどう考えるべきか

変額保険：マーケットリスク・アクチュアリーリスク　まったく異なる

# Decreasing Trends in Stock-Bond Correlations*

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# Decreasing Trends in Stock-Bond Correlations

**Abstract**

Previous research documents the existence of long-run trends in comovements among the stock, bond, and commodities markets. Following these findings, this paper examines possible trends in stock-bond return correlations. To this end, we introduce a trend component into a smooth transition regression (STR) model including the multiple transition variables of Aslanidis and Christiansen (2012). The results indicate the existence of significant decreasing trends in stock-bond correlations. In addition, although stock market volatility continues to be an important factor in stock-bond correlations, the short rate and yield spread become only marginally significant once we introduce the trend component. Our out-of-sample analysis also demonstrates that the STR model including the VIX and time trend as the transition variables dominates other models. Our finding of decreasing trends in stock-bond correlations can be considered a consequence of the decreasing effects of diversification and more intensive flight-to-quality behavior that have taken place in recent years.

JEL classification: C22, G15, G17

Key Words: flight-to-quality; diversification effect; smooth transition regressions

## 1 Introduction

Understanding time variations in stock-bond return correlations is one of the most important issues in finance because it has profound implications for asset allocation and risk management. Naturally, a number of studies examine the dynamics of stock-bond correlations and identify the economic factors driving their time series behavior. For instance, Li (2002) conducts a regression analysis to investigate the relationship between stock-bond correlations and macroeconomic variables, showing that unexpected inflation is the most important determinant of stock-bond correlations. Similarly, Ilmanen (2003) argues that stock-bond correlations are more likely to be negative when inflation is low and stock market volatility is high. Yang, Zhou, and Wang (2009) examine stock-bond correlations over the past 150 years using the smooth transition conditional correlation (STCC) model and find that higher stock-bond correlations tend to follow higher short rates and (to a lesser extent) higher inflation rates. In addition, Connolly, Stivers, and Sun (2005, 2007) identify the VIX stock market volatility index as an important determinant of stock-bond correlations. Furthermore, Aslanidis and Christiansen (2010, 2012) demonstrate that stock-bond correlations are explained mostly by short rates, yield spreads, and the VIX. On the other hand, Pastor and Stambaugh (2003) note that changes in stock-bond correlations depend on liquidity. Similarly, Baele, Bekaert, and Inghelbrecht (2010) find that macroeconomic fundamentals contribute little to explaining stock-bond correlations but that liquidity plays a more important role. Other related studies include Guidolin and Timmermann (2006); Bansal, Connolly, and Stivers (2010); and Viceira (2012).

A number of recent studies also investigate long-run trends in international financial markets. For instance, Christoffersen et al. (2012) examine copula correlations in international stock markets and find a significant increasing trend that can be explained by neither volatility nor other financial and macroeconomic variables. Similarly, Berben and Jansen (2005) and Okimoto (2011) report increasing dependence in major equity markets. In international bond markets, Kumar and Okimoto (2011) find an increasing trend in correlations among international long-term government bonds and a decreasing trend in correlations between short- and long-term government bonds within single countries. Existing trends in comovements are also documented in commodities markets. For example, Tang and Xiong (2012) show that the prices of non-energy commodity futures in the US have become increasingly correlated with oil prices. In addition, Ohashi and Okimoto (2013) find increasing trends in the excess comovements of commodities prices. Other related studies include Longin and Solnik (1995), Silvennoinen and Teräsvirta (2009), and Silvennoinen and Thorp (2013).

The main contribution of this paper is to provide new evidence of long-run decreasing trends in stock-bond correlations by extending the smooth transition regression (STR) model of Aslanidis and Christiansen (2012). Although a growing number of studies exploring long-run trends in international financial markets suggest that it is of interest to analyze possible trends in stock-bond correlations, none of the previously mentioned studies consider these types of trends. Thus, it is very instructive to investigate long-run trends in stock-bond correlations. Indeed, our results indicate that there is a significant decreasing trend in realized stock-bond correlations. More importantly, although stock market volatility continues to be an important factor for stock-bond correlations, other important financial variables, namely the short rates and spreads between long- and short-term interest rates, become only marginally significant once we introduce the decreasing trend. Our out-of-sample analysis also indicates that the STR model including the VIX and time trend as the transition variables dominates other models. Our finding of a decreasing trend in stock-bond correlations can be considered a consequence of the decreasing effects of diversification and more intensive flight-to-quality behavior that have taken place in recent years.

The remainder of the paper is organized as follows: Section 2 presents the model, while Section 3 conducts the empirical analysis and Section 4 provides the conclusion.

## 2 Smooth Transition Regression Model

The main purpose of this paper is to examine possible long-run trends in realized stock-bond return correlations. To this end, we employ the smooth-transition model that was developed by Teräsvirta (1994) in the AR model framework and later used to analyze the determinants of stock-bond correlations by, among others, Yang, Zhou, and Wang (2009) and Aslanidis and Christiansen (2012). The former authors model correlations as latent variables and analyze them using the STCC model, whereas the latter authors investigate the realized correlation based on the smooth transition regression (STR) model with multiple transition variables. We employ the latter approach in this paper because it considerably facilitates the examination of the determinants of the time series behavior of stock-bond correlations, as emphasized by Aslanidis and Christiansen (2012). In addition, many other studies, including Ilmanen (2003) and Connolly et al. (2005, 2007), examine realized correlations. In particular, we apply the STR model with multiple transition variables to the realized correlations, following Aslanidis and Christiansen (2012).

The STR model used by Aslanidis and Christiansen (2012) is given by

$$FRC_t = \rho_1\{1 - F(s_{t-1})\} + \rho_2 F(s_{t-1}) + \varepsilon_t \tag{1}$$

where $FRC_t$ is the Fisher transformation of the realized correlation, $RC_t$, namely

$$FRC_t = \frac{1}{2}\log\left(\frac{1 + RC_t}{1 - RC_t}\right), \tag{2}$$

converting the realized correlation into a continuous variable not bounded between $-1$ and 1.[1] $F(s_{t-1})$ in (1) is the logistic transition function, taking values between 0 and 1. If $F(s_{t-1}) = 0$, the average value of $FRC$ would be $\rho_1$ and if $F(s_{t-1}) = 1$, the average value of $FRC$ would be $\rho_2$. In this sense, $\rho_1$ and $\rho_2$ in (1) can be considered the average correlations in regimes 1 and 2, respectively.[2] Thus, the conditional mean of $FRC_t$ is modeled as the weighted average of the two correlation extremes; the weight is decided by $F(s_{t-1})$. $s_{t-1} = (s_{1,t-1}\ s_{2,t-1}\ \cdots\ s_{K,t-1})'$ is a $K \times 1$ vector of transition variables,[3] governing the transition between regimes 1 and 2. Specifically,

[1]As a realized correlation, Aslanidis and Christiansen (2012) use the weekly sample correlation calculated from five-minute high frequency stock and bond returns without demeaning, whereas we use monthly sample correlations based on daily data with demeaning.

[2]Specifically, $\rho_1$ is the average "Fisher-transformed correlation." In what follows, we simply refer to this as "correlation".

[3]In practice, all transition variables are standardized to have a mean of 0 and a variance of 1 as Aslanidis and Christiansen (2012).

$F(s_{t-1})$ is expressed as

$$F(s_{t-1}) = \frac{1}{1+\exp[-\gamma'(s_{t-1}-c)]}$$
$$= \frac{1}{1+\exp[-\gamma_1(s_{1,t-1}-c)+\cdots-\gamma_K(s_{K,t-1}-c)]}, \tag{3}$$

where $\gamma_k$ is assumed to be positive for at least one $k$ to identify the STR model with multiple transition variables. The location parameter $c$ decides the center of the transition, while the smoothness parameter vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_K)'$ specifies the speed of the transition. More precisely, the transition caused by the transition variable $s_{k,t-1}$ is abrupt for large values of $\gamma_k$ and gradual for small values of $\gamma_k$. One of the main advantages of the STR model is that it can detect detect, from the data, when and how any transitions occur in stock-bond correlations. In addition, the STR model can describe a wide variety of change patterns, depending on the parameters $c$ and $\gamma$, which can be estimated from the data. Thus, by estimating the STR model, we can estimate the best transition patterns in stock-bond correlations.

In contrast to Aslanidis and Christiansen (2012), we use time trends as one of the transition variables to capture long-run trends in stock-bond correlations, following Lin and Teräsvirta (1994). In this framework, the time-varying correlation $FRC_t$ changes smoothly from $\rho_1$ to $\rho_2$ with time, assuming that $\gamma_k$ for the time trend is positive. Thus, we can interpret $\rho_1$ as a correlation around the beginning of the sample and $\rho_2$ as correlation around the end of the sample. A similar model is applied to conditional correlations by, among others, Berben and Jansen (2005) and Kumar and Okimoto (2011), who examine trends in stock and bond markets, respectively. This paper differs from these studies by investigating possible trends in stock-bond return correlations.

One concern about STR model (1) is possible serial correlation in $FRC_t$. Aslanidis and Christiansen (2012) address the serial correlation of the error term by calculating the Newey-West standard errors. However, if $FRC_t$ itself has a serial correlation, this results in the inconsistent estimates of the correlation parameters. Indeed, a number of studies based on the dynamic conditional correlation (DCC) model of Engle (2002) suggest that the conditional correlations among financial returns are typically highly serially correlated. To address possible serial correlations in $FRC_t$, we modify STR model (1) by including the AR(1) term as follows:

$$FRC_t = \rho_1\{1 - F(s_{t-1})\} + \rho_2 F(s_{t-1}) + \phi FRC_{t-1} + \varepsilon_t. \tag{4}$$

In this STR model, $FRC_t$ can be expressed as the weighted sum of the correlations expected by the economic variables and the previous correlation level. Theoretically, this model is also relevant because economic conditions may not be reflected immediately due, in part, to slow reactions by and imperfect information available to market participants. Therefore, the correlation may be adjusted slowly from the previous level, as in STR model (4).

We estimate STR model (4) using the maximum likelihood estimation (MLE) method, assuming that $\varepsilon_t$ follows independently and is identically normally distributed. If the normal distribution assumption is inappropriate, the estimation can be considered to follow the nonlinear least squares method.

# 3 Empirical Analysis

## 3.1 Data

Our empirical analysis is based on monthly data, with the sample period lasting from January 1991 to May 2012. All data used in the analysis are obtained from DataStream. The analyzed countries are the United States (US), Germany (GER), and the United Kingdom (UK). Initially, we obtain daily data on futures contracts in the stock and bond markets of these three countries. Using the daily data, we obtain the realized stock-bond return correlations in each country for each month. We use futures on the S&P 500 (US), DAX (GER), and FTSE (UK) stock indices to calculate stock returns and each country's ten-year bond futures to calculate bond returns.

We also obtain the VIX, short rate, and yield spread as transition variables, following Aslanidis and Christiansen (2012), who demonstrate that these three variables are the most important transition variables for determining stock-bond correlation regimes. These three variables are also documented as important determinants of stock-bond correlations by many previous studies. For instance, Aslanidis and Christiansen (2010) find that these three variables are by far the most critical predictors of stock-bond correlations at their low and high quantiles. In addition, Connolly, Stivers, and Sun (2005, 2007) identify the VIX stock market volatility index a factor that influences stock-bond correlations, while Baele, Bekaert, and Inghelbrecht (2010) use the short rate as an important explanatory variable for stock-bond correlations. Furthermore, Viceira (2012) finds that short rates and yield spreads are the two most important predictors of the realized bond CAPM beta and the bond C-CAPM beta.

The VIX ($VIX$) is the volatility index for the Chicago Board of Options Exchange (CBOE) and is based on the volatility of options on the S&P 500 index. We use the US VIX for all countries due to the limited availability of VIX data for the two other examined countries.[4] The short rate ($R$) is the three-month Treasury bill rate from the secondary market for the US and the three-

[4] We confirm that the German and UK VIX indices are highly correlated with the US VIX, with a correlation that is greater than 0.8. We also confirm that we can obtain quantitatively similar results even if we use each country's VIX data with a shorter sample period.

month LIBOR rate for Germany and UK, while the yield spread ($SPR$) is defined as the ten-year constant maturity Treasury bond yield minus the short-rate for each country.

## 3.2 Benchmark Model Results

Our benchmark model is Aslanidis and Christiansen's (2012) preferred model, namely STR model (4), with $s_{t-1} = (VIX_{t-1}, R_{t-1}, SPR_{t-1})'$. We refer to this model Model 1 and its estimation results are presented in Table 1, in which several items are worth noting. First, the last two rows of the table report the results of a version of Teräsvirta's (1994) linearity test and Eitrheim and Teräsvirta's (1996) additive nonlinearity test. As can be seen, the linearity test rejects the null of linearity in favor of the STR alternative at the 1% significance level for all countries. In contrast, the additive nonlinearity test is not significant, meaning that the proposed model adequately captures all smooth transition regime-switching behavior in the data without additional regimes for all countries.

Second, the AR parameters $\phi$ are highly significant, with estimated values of 0.38, 0.34, and 0.25 for the US, GER, and the UK, respectively. In other words, our results indicate that stock-bond correlations change from the previous level toward the correlation level expected by economic variables with some serial correlation, which is not captured by Aslanidis and Christiansen's (2012) original model.

Third, the correlation parameters for regime 1 are significantly positive, with estimated values of 0.30, 0.38, and 0.44 for the US, GER, and the UK, respectively, while those for regime 2 are significantly negative, with respective values of $-0.32$, $-0.40$, and $-0.36$. In other words, there are two distinct regimes, one with positive average correlations and the other with negative average correlations. Thus, correlations change smoothly or rapidly from positive to negative or negative to positive, depending on the transition variables.

Finally, all three transition variables, the VIX, short rate, and yield spread, have statistically significant effects on the regime transition at the 5% significance level for all countries. These results are fairly consistent with those of Aslanidis and Christiansen (2012), who demonstrate that stock-bond correlations are explained mostly by these three variables using STR model (1) without the AR term. These three variables are also reported to be important determinants of stock-bond correlations by other studies. For instance, the VIX is identified as a predominant factor for stock-bond correlations by Connoly et al. (2005, 2007) and Bansal et al. (2010). In addition, Baele et al. (2010) use the short rate as an important explanatory variable for stock-bond correlations, while Yang, Zhou, and Wang (2009) find that higher stock-bond correlations tend to follow higher short rates. Furthermore, Viceira (2012) finds that the yield spread and the short rate are important predictors for the realized bond CAPM beta and bond C-CAPM beta, which can be regarded as a transformation of the stock-bond correlation.

To see more detailed information on the regime transitions for each variable, the transition functions of each variable are plotted in Figure 1, holding the other variables constant at their mean values of zero. As can be seen, there is little difference across countries in terms of short rates and yield spreads and the correlation regime changes rather rapidly from the negative regime to the positive regime as these variables get larger. For instance, if the short rate is lower than the average by one standard deviation, the transition function takes a value greater than 0.97, meaning that the weight of the negative correlation regime is greater than 97%. More specifically, if the short rate is lower than the average value by one standard deviation, the average correlation is less than $-0.30$, $-0.39$, and $-0.35$ for the US, GER, and the UK, respectively. On the other hand, if the short rate is higher than the average value plus one standard deviation, the weight of negative regime becomes less than 0.04, making the average correlation more than 0.28 for all countries. Similarly, if the yield spread is lower (larger) than the average value by one standard deviation, the transition function is greater (less) than 0.90 (0.11), with an average correlation of less than $-0.26$ (greater than 0.18) for all countries. Since larger yield spreads and short rates are usually associated with better macroeconomic conditions, the results indicate that stock-bond correlations tend to be positive when the economy is booming. In other words, when the economy is in recession, stock-bond correlations have a tendency to be negative. This is arguably consistent with flight-to-quality behavior because investors do not want to take many risks when economic conditions are not good.

The VIX transition function also demonstrates flight-to-quality behavior. For the US and GER, the VIX transition function indicates that the correlation regime changes relatively smoothly from the negative regime to the positive regime as the standardized VIX changes from $-3$ to 3. The UK VIX transition function indicates slower changes in the correlation regime but still suggests that a higher VIX tends to be associated with negative stock-bond correlations. Thus, the results demonstrate that when the VIX is high or there is much uncertainty in the market, investors try to escape from risks, making stock-bond correlations negative.

Finally, the time series of the estimated correlations for Model 1 together with the actual realized correlations for each country are plotted in Panel (a) of Figures 2-4 to indicate goodness of fit. As can be seen, the estimated correlation fits the actual correlation quite well for all countries. More specifically, Model 1 successfully captures the tendency for there to be positive correlations before 2000 and negative correlations after 2000 because the correlation regimes tend to be identified as the positive regime before 2000 and the negative regime after 2000.

In sum, the results of Model 1 indicate that the VIX, short rate, and yield spread are important determinants of stock-bond correlation regimes for all countries, which is consistent with previous studies such as Aslanidis and Christiansen (2012), who estimated a similar model for the US. In addition, we demonstrate the significance of including the AR(1) to allow for smooth adjustments in correlation regimes, in contrast with Aslanidis and Christiansen (2012). Although the performance of Model 1 is quite satisfactory, it is possible to improve Model 1 by including other variables. In particular, recent studies find long-run correlation trends in international financial markets, suggesting that we can modify Model 1 by introducing a time trend component; this is examined in next subsection.

## 3.3 Introduction of Time Trend Component

The results of Model 1 are fairly consistent with previous studies examining the dynamics of stock-bond correlations. On the other hand, the another previous studies suggest the existence of long-run correlation trends in international financial markets. For instance, Christoffersen et al. (2012) examine copula correlations in international stock markets and find a significant increasing trend in the comovements of international stock returns that can be explained by neither volatility nor other financial and macroeconomic variables. In addition, Kumar and Okimoto (2011) find an increasing trend in correlations between international long-term government bonds and decreasing trends in correlations between the short- and long-term government bonds within single countries. Furthermore, Tang and Xiong (2012) document increasing correlations of commodities returns with crude oil after 2004. It is therefore of interest to analyze possible trends in stock-bond correlations by estimating STR model (4) including time ($T$) as well the VIX, short-rate, and spread as transition variables (Model 2). Thus, the vector of transition variables for Model 2 is defined as $s_{t-1} = (VIX_{t-1}, R_{t-1}, SPR_{t-1}, T_t)'$.[5]

Table 2 reports the estimation results for Model 2. As can be seen, the results suggest that the basic structure of Model 2 is reasonably similar to that of Model 1. Specifically, the linearity and additive nonlinearity tests documented in the last two rows of Table 2 show that the two-state STR model is preferred to the linear model without regime changes and the three-state STR model with an additional correlation regime. In addition, Model 2 indicates the existence of two distinct correlation regimes, with a negative average correlation for one regime and a positive average correlation for the other, as in Model 1. Furthermore, the AR term is significant at least at the 10% significance level for the US and GER, suggesting smooth adjustments in stock-bond correlations in these countries.

[5]Since $T$ is a non-random predetermined variable, we use $T_t$ instead of $T_{t-1}$ as a transition variable.

Although the basic structures of Models 1 and 2 are quite similar, there are important differences in the determinants of their stock-bond correlation regimes. In particular, the estimation results of Model 2 indicate that the time trend component is highly significant for all countries, suggesting that Model 1 omits an important factor of stock-bond correlations. More specifically, the time trend component coefficient estimates are significantly positive for all countries, meaning that there is a decreasing trend in stock-bond correlations. To see this more clearly, we plot the time trend for the correlations estimated through Model 2 in Panel (a) of Figure 5. As can be seen, the stock-bond correlations for all countries have clear decreasing trends, with a rapid decrease between the late 1990s and the early 2000s, reaching an average of $-0.42$ by the end of sample period in May 2012. Our finding of the existence of a time trend in correlations between financial assets is completely in line with recent studies. For instance, Berben and Jansen (2005) and Christoffersen (2012) document increasing correlations in the major equity markets. Similarly, Kumar and Okimoto (2011) find an increasing trend in correlations between international long-term government bonds and decreasing trends in correlations between a single country's short- and long-term government bonds.

Another important difference between Models 1 and 2 is the significance of the short rate and yield spread in determining the stock-bond correlation regime. Although the VIX remains an important factor in determining stock-bond correlations, the short rate and yield spread become less important in Model 2. Specifically, neither of these measures are significant for the US, while only one of them is significant for GER and the UK. In addition, the the short rate coefficient for GER is significantly positive instead of negative, making interpretation of the result rather difficult. The results are in contrast with the findings of the previously mentioned studies examining the determinants of stock-bond correlations without a time trend component. Thus, our results demonstrate that some of the important factors suggested by previous studies are not as relevant once we consider possible decreasing trends in stock-bond correlations.

To compare the goodness of fit of Models 1 and 2, we plot the time series of the correlations estimated through Model 2 together with the actual realized correlations for each country in Panel (b) of Figures 2-4. As can be seen, the correlations estimated through Models 1 and 2 are similar to each other and do not differ much over the sample. Thus, they qualitatively have the same power in illustrating the time series behavior of stock-bond correlations.

We can compare the goodness of fit of Models 1 and 2 more formally using the information criteria reported in Table 3, namely the Schwartz information criterion (SIC) and Akaike information criterion (AIC). Although the AIC favors Model 2 for GER and the UK, the SIC prefers Model 1 to Model 2 for all countries. Thus, in terms of the in-sample fit, our results are somewhat inconclusive.

To make a more comprehensive comparison between Models 1 and 2, we conduct an out-of-sample forecast evaluation as follows. First, we estimate both Models 1 and 2 using data from February 1991 to January 2001 and evaluate the terminal one-month-ahead forecast error based on the estimation results. The data are then updated by one month, and the terminal one-month-ahead forecast error is re-calculated from the updated sample (specifically, from March 1991 to February 2001). This procedure is repeated until reaching one month before the end of the sample period, namely April 2012. Finally, we calculate the root-mean-squared forecast errors (RMSE) and mean absolute error (MAE) using the obtained time series of one-month-ahead forecast errors. The third and fourth rows of Table 4 report the RMSE and MAE values for Models 1 and 2. As can be seen, the RMSE and MAE values of Model 2 are smaller than those of Model 1 for GER, while Model 1 exhibits better out-of-sample performance than Model 2 for other two countries.

Overall, our model comparison results show that Model 2 is not necessarily a better model than Model 1, although the time trend component is highly significant. One possible explanation for this result is the weak significance of the short rate and yield spread in Model 2, as mentioned. Indeed, neither of these factors are significant for US, while only one of them is significant for GER and UK. Thus, we might be able to improve the model by excluding these variables. To examine this possibility, we will consider a more parsimonious model in next subsection.

## 3.4 Results with Selected Transition Variables

Our results for Model 2 indicate that the short rate and yield spread become less important determinants of stock-bond correlations if decreasing trends in stock-bond correlations are taken into consideration. To illustrate this point more clearly, we estimate a more parsimonious STR model (4) that includes only VIX and time as the transition variables (Model 3).

The estimation results for Model 3 are shown in Table 3. As can be seen, the estimation results are essentially same as those of Model 2. The two-state STR model with a negative average correlation for one regime and a positive average correlation for the other regime is preferred to the linear model without regime changes and the three-state STR model. In addition, the AR term is highly significant for the US and GER, suggesting that the stock-bond correlations of these countries change slowly from the previous level toward the correlation level expected by economic variables. Furthermore, the VIX is significantly positive for all countries. Thus, the correlation regime changes from a positive to a negative regime when the VIX is high. Finally, the estimated time trend component is also significantly positive for all countries, meaning that stock-bond correlations tend to be in the negative regime in more recent periods. The decreasing trend can be confirmed visually from the estimated time trend component of stock-bond correlation depicted in Panel (b) of Figure 5. As can be seen from the figure, stock-bond correlations in all countries exhibit clear decreasing trends, with a rapid decrease from an average correlation of over 0.2 in the beginning of 1999 to an average correlation lower than $-0.2$ at the end of 2003, reaching an average of $-0.42$ around the end of the sample period in May 2012.

We also plot the time series of the estimated correlation for Model 3 together with the actual realized correlation for each country in Panel (c) of Figures 2-4 to graphically illustrate the performance of Model 3. As can be seen, the estimated correlations of Model 3 are quite similar to those of other models and do not differ much over the sample, suggesting that all models have the same qualitative explanatory power over stock-bond correlation behavior. Given that Model 3 has only two transition variables, this arguably indicates the superiority of Model 3 over the other two models. We can confirm this point more formally using the SIC and AIC reported in Table 3. As can be seen, Model 3 has the smallest SIC and AIC values for all countries, meaning that Model 3 is the best among the three models in terms of in-sample fit.

We additionally compare the out-of-sample performance of Model 3 and the other two models by conducting the same out-of-sample forecast evaluation as before. The results reported in Table 4 indicate that Model 3 exhibits the best out-of-sample performance for all countries, regardless of the employed performance measure.

In sum, our results are clear: Model 3 is the best among the three models, meaning that transitions between correlation regimes can be described sufficiently well by the VIX and time trend components. In other words, we demonstrate the possibility that the short rate and yield spread are not important factors in relation to stock-bond correlation regimes, in great contrast to previous studies such as Aslanidis and Christiansen (2012). Thus, flight-to-quality behavior is not strongly related with economic conditions, measured by short rates and yield spreads, but is associated with market uncertainty, as captured by the VIX. In addition, flight-to-quality behavior has become stronger in more recent years, resulting in decreasing trends in stock-bond correlations.

A possible explanation for this trend in flight-to-quality behavior is the recent increasing trend in correlations in international equity markets, which is documented by Christoffersen, et al. (2012), among others. Specifically, they emphasize that benefits from international diversification have decreased over time and this decrease has been especially drastic among developed markets, such as those examined in this study. In addition, Berben and Jansen (2005) show that correlations among the GER, UK, and US stock markets have doubled between 1980 and 2000. Similarly, Silvennoinen and Teräsvirta (2009) show that stock returns within and across European and Asian markets exhibit a clear upward shift in the level of correlations between 1998 and 2003, which corresponds to the timing of the rapid decrease in the estimated time trend of stock-bond correlations from our models. Thus, benefits from international diversification seem to begin disappearing after 2000. In this case, the investors who allocated their money into the equity markets of those countries have been exposed to higher risks of simultaneous drops in stock prices in recent years. As a consequence, they have more recently needed to make greater use of bond markets to control their risk exposure, producing the decreasing trend in stock-bond correlations. Indeed, the beginning of the integration of international equity markets and the beginning of decreases in stock-bond correlations appear to occur around the same time.

In addition to integration in equity markets, increasing correlations are observed in other markets as well. For instance, Kumar and Okimoto (2011) show that long-term government bond markets have become more integrated since the late 1990s, while Silvennoinen and Thorp (2013) find that correlations among stock, bond, and commodity future returns greatly increased around the early 2000s. Similarly, Tang and Xiong (2012) document increasing correlations of non-energy commodity with crude oil after 2004. These phenomena further diminish the effects of diversification in international financial markets, making investors diversify risks through bond markets. This phenomenon induces a rebalancing, particularly with from stocks to bonds.

Fleming, Kirby, and Ostdiek (1998) and Kodres and Pritsker (2002) study how cross-market hedging theoretically influences asset pricing. Specifically, Fleming, Kriby, and Ostdiek (1998) demonstrate that information linkages in stock and bond markets may be greater if cross-market hedging effects are considered within daily returns. In addition, Kodres and Pritsker (2002) show that a shock in one asset market may generate cross-market rebalancing, which influences prices in non-shocked asset markets. Since the disappearance of diversification effects produces investment behavior involving rebalancing from stocks to bonds, correlations between stocks and bonds tend to be negative, which can be captured by a trend variable, as indicated by our results.

# 4 Conclusion

In this paper, we investigated the existence of long-run trends in realized stock-bond return correlations. To this end, we introduce a trend component into the smooth transition regression (STR) model with the multiple transition variables of Aslanidis and Christiansen (2012). In addition, we analyzed not only the US, but also Germany and the UK, to conduct a more comprehensive examination. The results indicated the existence of a significant decreasing trend in stock-bond correlations for all countries.

Since a number of studies based on the dynamic conditional correlation (DCC) model of Engle (2002) suggest that conditional correlations between financial returns are typically highly serially correlated, we extended the STR model of Aslanidis and Christiansen (2012) by including the AR(1) term. The AR parameter estimates are highly significant for all countries. Thus, our results demonstrated that stock-bond correlations change slowly from the previous level toward the correlation level expected by economic variables, which is not captured by the original model of Aslanidis and Christiansen (2012).

In the case of transition variables, we examined three variables, namely the VIX, short rate, and yield spread, which have been identified by previous studies as arguably three of the most important factors. All three transition variables have statistically significant effects on regime transitions for all countries in our extended model. The results are fairly consistent with those of previous studies, particularly Aslanidis and Christiansen (2012). However, once we introduce the trend component, although the VIX remains an important factor for stock-bond correlations, the short rate and yield spread become only marginally significant. Indeed, our in-sample analysis suggested that the STR model including the VIX and time trend as the transition variables is the best model based on the SIC and AIC, meaning that the transition of stock-bond correlation regimes can be described sufficiently well by the VIX and time trend components. In addition, our out-of-sample analysis also demonstrated that the STR model with the VIX and time trend as the transition variables dominates other models.

Previous studies document the existence of long-run trends in comovements in the stock, bond, and commodities markets, suggesting that benefits from international diversification have recently been disappearing. Therefore, investors have been exposed to higher risks of simultaneous drops in stock prices in recent years. As a consequence, they have needed to make greater use of bond markets to control their risk exposure, producing the decreasing trend in stock-bond correlations. Interestingly, the beginning of the integration of international equity markets suggested by several previous studies and the beginning of decreases in stock-bond correlations appear to occur around the same time. Thus, our finding of a decreasing trend in stock-bond correlations can be considered a consequence of decreasing diversification effects and more intensive flight-to-quality behavior in recent years.

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Table 1: Estimation results of the benchmark model (Model 1)

| | US | | GER | | UK | |
|---|---|---|---|---|---|---|
| | Coef | St. err | Coef | St. err | Coef | St. err |
| $\rho_1$ | 0.298*** | 0.101 | 0.378** | 0.164 | 0.437*** | 0.055 |
| $\rho_2$ | -0.321*** | 0.129 | -0.404*** | 0.147 | -0.360*** | 0.038 |
| $\phi$ | 0.380*** | 0.090 | 0.342** | 0.134 | 0.249*** | 0.080 |
| VIX | 1.370*** | 0.206 | 1.308*** | 0.099 | 0.537*** | 0.103 |
| R | -3.414*** | 1.018 | -3.968*** | 0.528 | -3.824*** | 0.097 |
| SPR | -2.201*** | 0.673 | -2.839*** | 0.610 | -2.476*** | 0.219 |
| c | 0.046 | 0.095 | 0.062 | 0.208 | -0.007 | 0.077 |
| Log-likelihood | -248.86 | | -250.95 | | -248.34 | |
| Linearity test | 12.3*** | | 24.44*** | | 16.55*** | |
| Additive nonlinearity test | 0.22 | | 0.73 | | 0.20 | |

Note: the table shows the estimation results of the STR Model 1 with transition variables; VIX index (VIX), short rate (R), yield spread (SPR). */**/*** indicates that the variable is significant at the 10%/5%/1% level of significance, respectively. Linearity test reports the LM-type statistic of null of no STR-type nonlinearity. Additive non-linearity shows the LM-Type statistic of null on no remaining STR-type nonlinearity.

Table 2: Estimation results of the model with time trend component (Model 2)

| | US | | GER | | UK | |
|---|---|---|---|---|---|---|
| | Coef | St. err | Coef | St. err | Coef | St. err |
| $\rho_1$ | 0.297** | 0.140 | 0.630*** | 0.052 | 0.502*** | 0.117 |
| $\rho_2$ | -0.368*** | 0.099 | -0.580*** | 0.027 | -0.440*** | 0.075 |
| $\phi$ | 0.346* | 0.192 | 0.140*** | 0.028 | 0.156 | 0.105 |
| VIX | 1.925*** | 0.616 | 1.142*** | 0.083 | 1.163*** | 0.354 |
| R | -0.576 | 0.461 | 1.323*** | 0.039 | 0.159 | 0.140 |
| SPR | -0.294 | 0.672 | 0.051 | 0.049 | -0.450*** | 0.161 |
| T | 2.571*** | 0.943 | 2.804*** | 0.010 | 2.725*** | 0.311 |
| c | 0.071 | 0.165 | -0.144*** | 0.054 | -0.065 | 0.158 |
| Log-likelihood | -248.23 | | -248.25 | | -247.29 | |
| Linearity test | 10.95*** | | 24.26*** | | 21.54*** | |
| Additive nonlinearity tes | 1.28 | | 2.55 | | 0.09 | |

Note: the table shows the estimation results of the STR Model 1 with transition variables; VIX index (VIX), short rate (R), yield spread (SPR), time trend (T). */**/*** indicates that the variable is significant at the 10%/5%/1% level of significance, respectively. Linearity test reports the LM-type statistic of null of no STR-type nonlinearity. Additive non-linearity shows the LM-Type statistic of null on no remaining STR-type nonlinearity.

Table 3: Results of in-sample comparison

| | US | | GER | | UK | |
|---|---|---|---|---|---|---|
| | AIC | SIC | AIC | SIC | AIC | SIC |
| Model 1 | 511.72 | 536.54 | 515.90 | 540.71 | 510.68 | 535.50 |
| Model 2 | 512.46 | 540.82 | 512.51 | 540.87 | 510.58 | 538.95 |
| Model 3 | 508.54 | 529.81 | 509.30 | 530.58 | 507.01 | 528.28 |

Note: the table reports the AIC and SIC for STR Models 1-3 to compare in-sample performance.

Table 4: Results of out-of-sample comparison

| | US | | GER | | UK | |
|---|---|---|---|---|---|---|
| | RMSE | MAE | RMSE | MAE | RMSE | MAE |
| Model 1 | 0.201 | 0.155 | 0.322 | 0.257 | 0.259 | 0.212 |
| Model 2 | 0.203 | 0.161 | 0.297 | 0.231 | 0.274 | 0.221 |
| Model 3 | 0.174 | 0.136 | 0.296 | 0.231 | 0.241 | 0.199 |

Notes: the table reports the out-of-sample RMSE and MAE for STR Models 1-3. The forecast horizon is 1 month and the forecast period is 2000/12-2012/05.

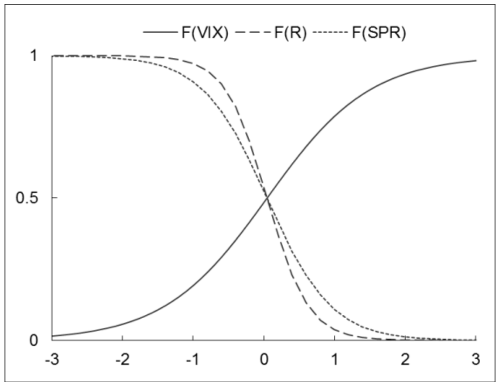
Table 5: Estimation results of the parsimonious model (Model 3)

| | US | | GER | | UK | |
|---|---|---|---|---|---|---|
| | Coef | St. err | Coef | St. err | Coef | St. err |
| $\rho_1$ | 0.289*** | 0.001 | 0.459*** | 0.002 | 0.483*** | 0.185 |
| $\rho_2$ | -0.363*** | 0.002 | -0.570*** | 0.006 | -0.419** | 0.173 |
| $\phi$ | 0.359*** | 0.001 | 0.136*** | 0.005 | 0.173 | 0.192 |
| VIX | 1.983*** | 0.003 | 1.901*** | 0.009 | 1.373*** | 0.345 |
| T | 2.959*** | 0.003 | 3.315*** | 0.095 | 2.808*** | 0.675 |
| c | 0.068* | 0.041 | 0.005 | 0.067 | -0.106 | 0.192 |
| LLF | -248.27 | | -248.65 | | -247.51 | |
| Linearity test | 21.33*** | | 36.88*** | | 38.87*** | |
| Additive nonlinearity test | 1.25 | | 0.02 | | 0.61 | |

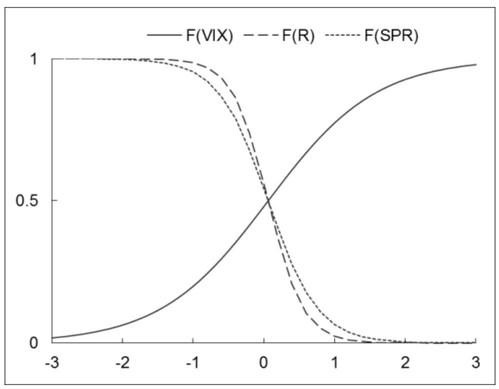
Note: the table shows STR Model 3 with transition variables; VIX index (VIX), Time Trend (T). */**/*** indicates that the variable is significant at the 10%/5%/1% level of significance, respectively. Linearity test reports the LM-type statistic of null of no STR-type nonlinearity. Additive non-linearity shows the LM-Type statistic of null on no remaining STR-type nonlinearity.

Figure 1: Estimated transition function

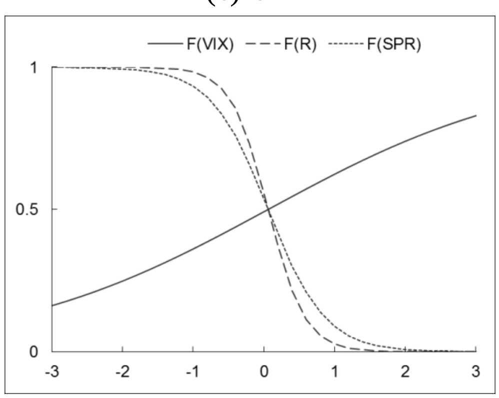
(a) US




(b) GER



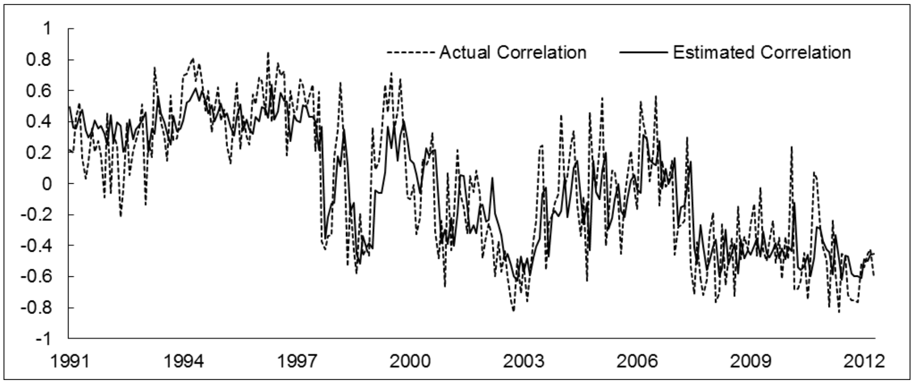

(c) UK



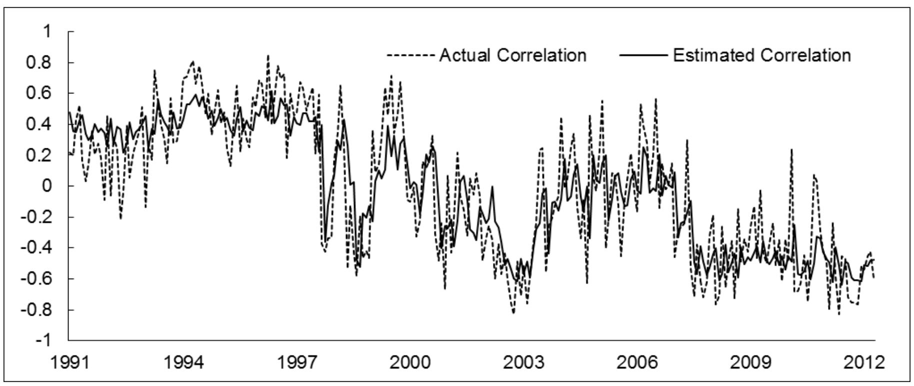

Notes: the graph shows the estimated transition function of model1 against each of the transition variables holding the other transition variables constant at their sample mean. The transition variables are VIX index (VIX), short rate (R ) ,and yield spread (SPR).

Figure 2: Estimated stock-bond correlation for US

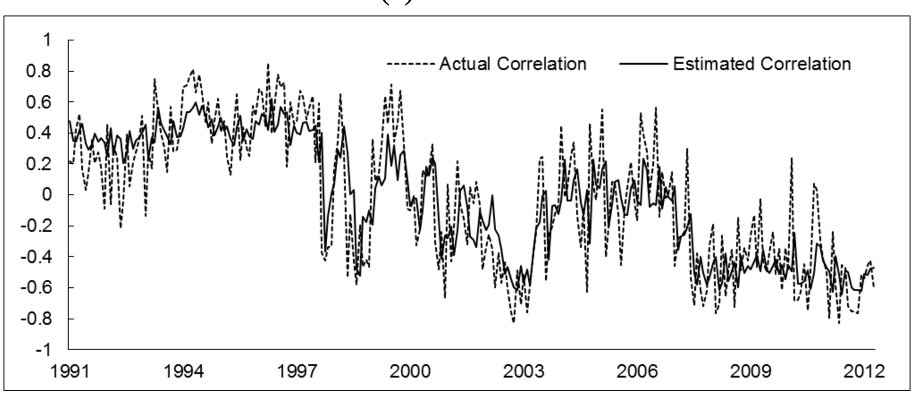
(a) Model 1




(b) Model 2



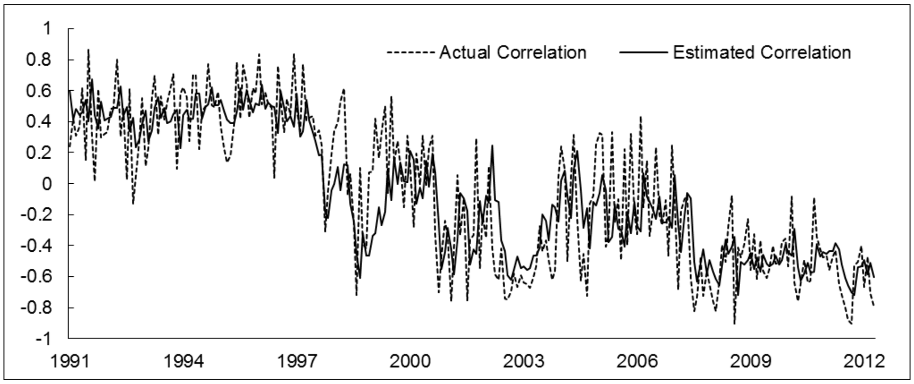

(c) Model 3



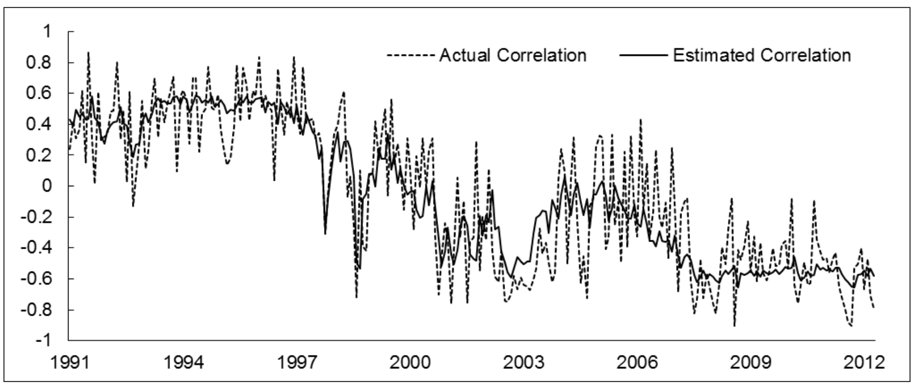

Notes: the graph shows the time series of the actual and estimated stock-bond correlation for Models 1-3 for US.

Figure 3: Estimated stock-bond correlation for GER

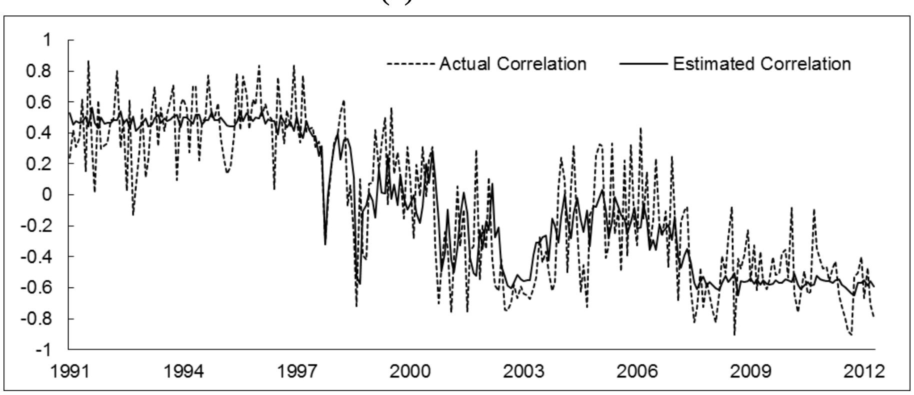
(a) Model 1




(b) Model 2



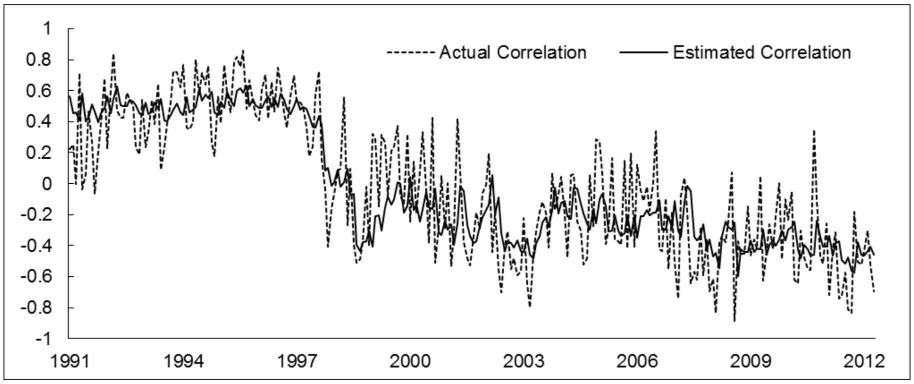

(c) Model 3



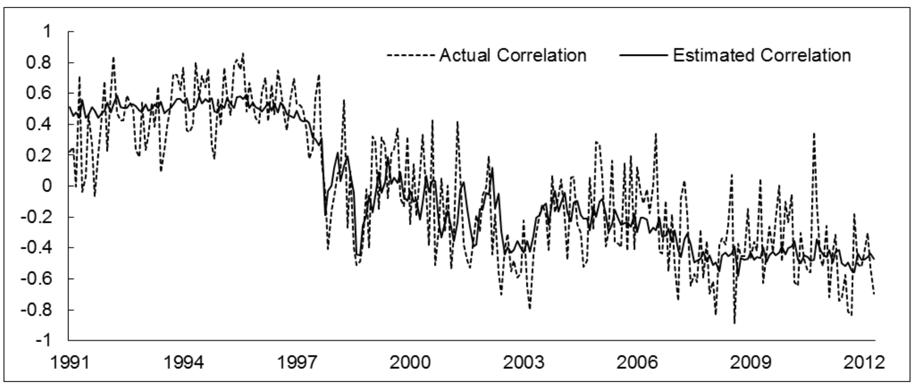

Notes: the graph shows the time series of the actual and estimated stock-bond correlation for Models 1-3 for GER.

Figure 4: Estimated stock-bond correlation for UK

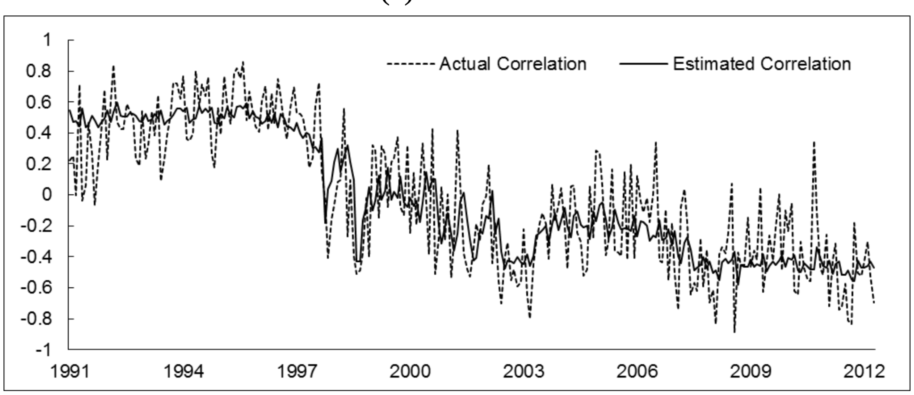
(a) Model 1




(b) Model 2



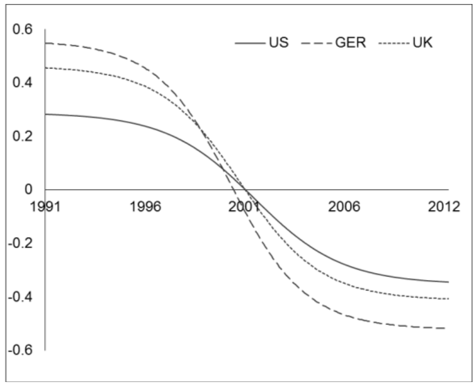

(c) Model 3



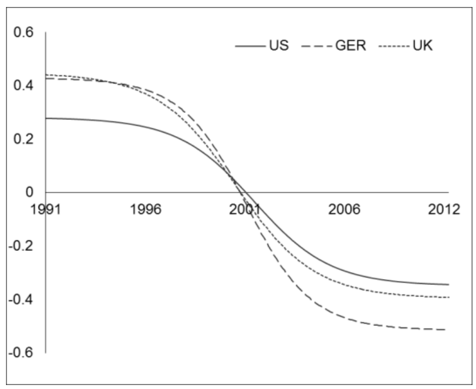

Notes: the graph shows the time series of the actual and estimated stock-bond correlation for Models 1-3 for UK.

Figure 5: Estimated time trend component in the stock-bond correlation

(a) Model 2




(b) Model 3




Note: the graph shows the time series of the estimated time trend component in the stock-bond correlation for Models 2 and 3.

Decreasing Trends in Stock-Bond Correlations

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December 2013

## Motivations and Main Results

**Motivations**

1. Stock-bond return correlations have profound implications on
   (a) Asset allocation
   (b) Risk management
2. Understanding the stock-bond correlations might not be easy due to the time variation of the correlation
3. Identifying the economic factors driving its time series behavior is one of the most important issues

4. Identified determinants of stock-bond correlations
   (a) Li (2002): (unexpected) inflation
   (b) Ilmanen (2003): inflation and stock market volatility
   (c) Yang, Zhou, and Wang (2009): short-rate and inflation
   (d) Connolly, Stivers, and Sun (2005, 2007): VIX
   (e) Aslanidis and Christiansen (2010, 2012): short-rate, yields spread, VIX
   (f) Pastor and Stambaugh (2003): liquidity
   (g) Baele, Bekaert, and Inghelbrecht (2010): liquidity



5. Long-run trends in international financial markets
   (a) Christoffersen et al. (2012)
      i. Find a significant increasing trend in correlations in international equity markets
      ii. Trend is much lower for emerging markets
      iii. Confirm that trend can be explained by neither volatility nor other financial and macroeconomic variables
   (b) Berben and Jansen (2005): international equity markets
   (c) Okimoto (2011): international equity markets
   (d) Kumar and Okimoto (2011)
      i. Find an increasing trend in correlations among international long-term government bonds
      ii. Detect a decreasing trend in correlations between short- and long-term government bonds within single countries

(e) Tang and Xiong

i. There was a significant and increasing trend in return correlations of non-energy commodities with oil after 2004

ii. Increasing trend is significantly stronger for indexed commodities (listed in either the SP-GSCI or DJ-UBS index) than for off-indexed commodities

(f) Silvennoinen and Thorp (2013): S&P500 and commodity future returns and returns to the majority commodity futures have increased

(g) Ohashi and Okimoto (2013): Excess comovements of commodities prices

(h) Few studies consider the possible trends in stock-bond correlations



**Main Results**

1. Examine the possible trend in stock-bond correlation
2. Extend Aslanidis and Christiansen (2012) in several ways

   (a) Treat serial correlations in stock-bond correlations explicitly

   (b) Introduce a time-trend component in stock-bond correlations

   (c) Examine Germany (GER) and UK as well as US

3. Find a significant decreasing trend in stock-bond correlations
4. Short rates and yield spreads become only marginally significant once we introduce the decreasing trend
5. STR model including the VIX and time trend as the transition variables dominates other models
6. Can be considered a consequence of the decreasing effects of diversification and more intensive flight-to-quality behavior

## Related Literature

**Aslanidis and Christiansen (2012)**

1. Explores the time variation in the stock-bond correlation using high-frequency data
2. Consider the smooth transition regression (STR) model with multiple transition variables
$$FRC_t = \rho_1\{1 - F(s_{t-1})\} + \rho_2 F(s_{t-1}) + \varepsilon_t,$$
where $FRC_t$ is the Fisher transformation of the realized correlation
3. Examined transition variables: VIX, short-rate, yield spread, stock return, bond return, inflation, GDP growth
4. Detect one positive and one negative correlation regime systematically related to movements in financial and to a minor extent macroeconomic transition variables
5. Conclude that the short rate, the yield spread, and the VIX are the most important factors



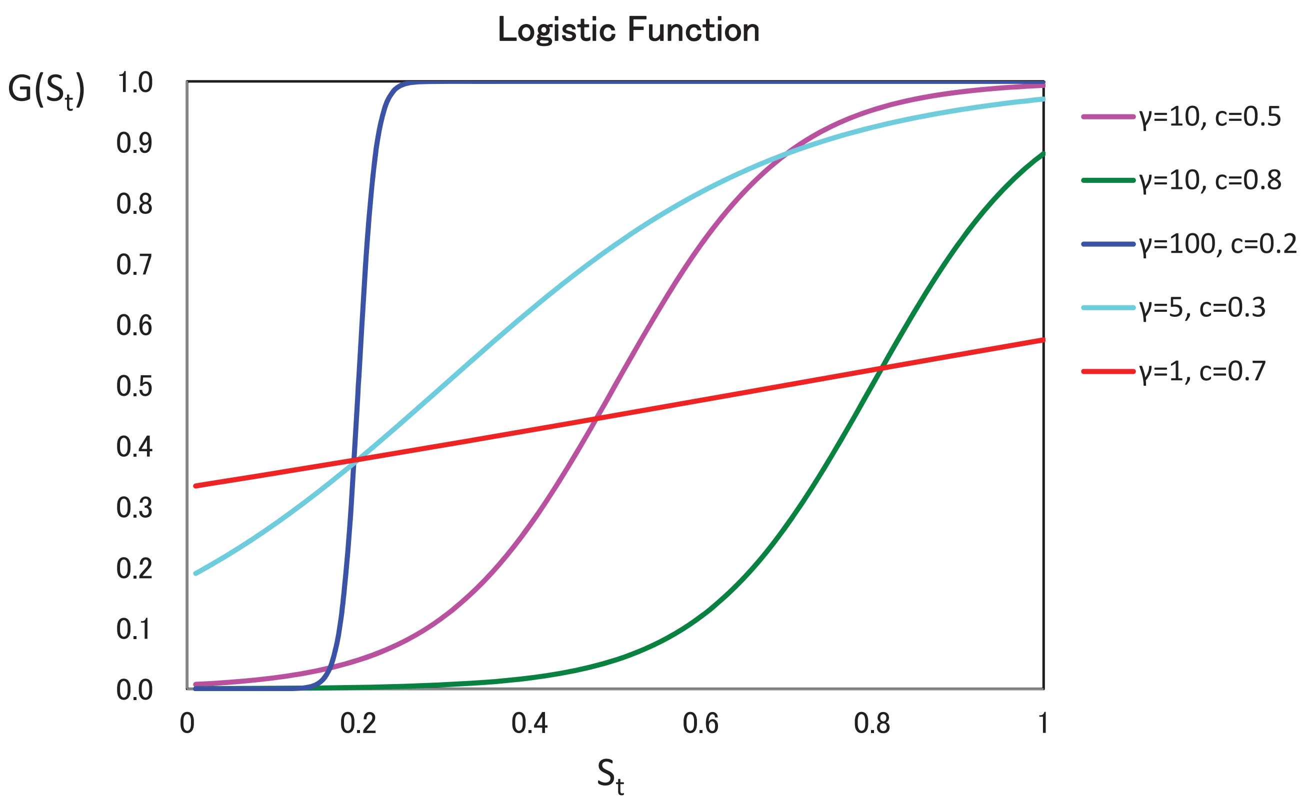
## Methodology

**STR model**

1. Developed by Teräsvirta (1994) in the AR framework
2. STR Model for $FRC_t$
$$FRC_t = \rho_1\{1 - F(s_{t-1})\} + \rho_2 F(s_{t-1}) + \varepsilon_t$$
3. One of the regime switching models
   (a) Regime 1: $F = 0 \Longrightarrow E(FRC_t) = \rho_1$
   (b) Regime 2: $F = 1 \Longrightarrow E(FRC_t) = \rho_2$
4. Regime transition is modeled by a logistic transition function $F$
$$F(s_{t-1}; c, \gamma) = \frac{1}{1 + \exp(-\gamma(s_{t-1} - c))}, \quad \gamma > 0$$
   (a) $s_t$ : Transition variable
   (b) $c$: Location parameter
   (c) $\gamma$: Smoothness parameter

5. $F$ increases monotonically in $s_{t-1}$ from 0 to 1
   (a) $\rho_1$: conditional mean of FRC when $s_{t-1}$ is small
   (b) $\rho_2$: conditional mean of FRC when $s_{t-1}$ is large
6. Typical choice of a transition variable
   (a) $s_{t-1} = VIX_{t-1}$
      i. $\rho_1$: conditional mean of FRC when $VIX_{t-1}$ is small
      ii. $\rho_2$: conditional mean of FRC when $VIX_{t-1}$ is large
   (b) $s_{t-1} = t/T$
      i. $\rho_1$: value of FRC around the beginning of the sample
      ii. $\rho_2$: value of FRC around the end of the sample
7. Can capture dominant long-run trends by adopting $s_t = t/T$ as one the transition variables (Lin and Teräsvirta, 1994)
8. Can describe a wide variety of patterns of change depending on the values of $\gamma$, $c$





9. Transition variable can be a vector of variables

$$F(s_{t-1}) = \frac{1}{1+\exp[-\gamma'(s_{t-1}-c)]}$$
$$= \frac{1}{1+\exp[-\gamma_1(s_{1,t-1}-c)+\cdots-\gamma_K(s_{K,t-1}-c)]}$$

10. All transition variables are standardized to have a mean of 0 and a variance of 1
11. Treat serial correlations in stock-bond correlations explicitly by including the AR term

$$FRC_t = \rho_1\{1-F(s_{t-1})\} + \rho_2 F(s_{t-1}) + \phi FRC_{t-1} + \varepsilon_t.$$



## Test of linearity against the STR model

1. STR model: $FRC_t = \rho_1\{1-F(s_{t-1})\} + \rho_2 F(s_{t-1}) + \varepsilon_t$
2. Interesting to test linearity or the null of $H_0 : \rho_1 = \rho_2$
3. Cannot use the standard $F$-test due to the unidentified parameters $\gamma$ and $c$ under the null
4. Luukkonen, Saikkonen, and Teräsvirta (1988) propose a simple test for the STR model with the logistic transition function
5. Derive auxiliary regression model by replacing $F$ with a first order Taylor expansion around $\gamma = 0$

$$FRC_t = \beta_0 + \beta_1 s_{1,t-1} + \beta_2 s_{2,t-1} + \cdots + \beta_K s_{K,t-1} + e_t$$

6. $H_0 : \rho_1 = \rho_2$ is equivalent to $H_0' : \beta_1 = \cdots = \beta_K$
7. $H_0' : \beta_1 = \cdots = \beta_K$ can be tested by the standard $F$ test
8. Can test the additive nonlinearity (i.e. two state v.s. three state) based on similar idea (Eitrheim and Teräsvirta, 1996)

## Empirical Analysis

**Data**

1. Sample period: from January 1991 to May 2012
2. Analyzed countries: GER, UK, US
3. Collect daily data on futures contracts in the stock and bond markets
4. Stock: S&P 500 (US), DAX (GER), and FTSE (UK) stock index futures
5. Bond: each country's ten-year bond futures
6. Calculate the Fisher transformation of monthly sample stock-bond return correlation
7. Obtain the VIX, short rate, and yield spread as transition variables
8. Use the US VIX for all countries due to the limited availability of VIX data for the two other examined countries



**Benchmark model results**

1. Model 1: STR model with $s_{t-1} = (VIX_{t-1}, R_{t-1}, SPR_{t-1})'$
2. Aslanidis and Christiansen's (2012) preferred model
3. Linearity test rejects the null of linearity in favor of the STR alternative at the 1% significance level for all countries
4. Additive nonlinearity test is not significant for all countries
5. Two-state model adequately captures all smooth transition regime-switching behavior in the data
6. AR parameters $\phi$ are highly significant
7. There are two distinct regimes, one with positive average correlations and the other with negative average correlations
8. All three transition variables have statistically significant effects on the regime transition
9. Mostly consistent with Aslanidis and Christiansen's (2012)

Table 1: Estimation results of the benchmark model (Model 1)

| | US | | GER | | UK | |
|---|---|---|---|---|---|---|
| | Coef | St. err | Coef | St. err | Coef | St. err |
| $\rho_1$ | 0.298*** | 0.101 | 0.378** | 0.164 | 0.437*** | 0.055 |
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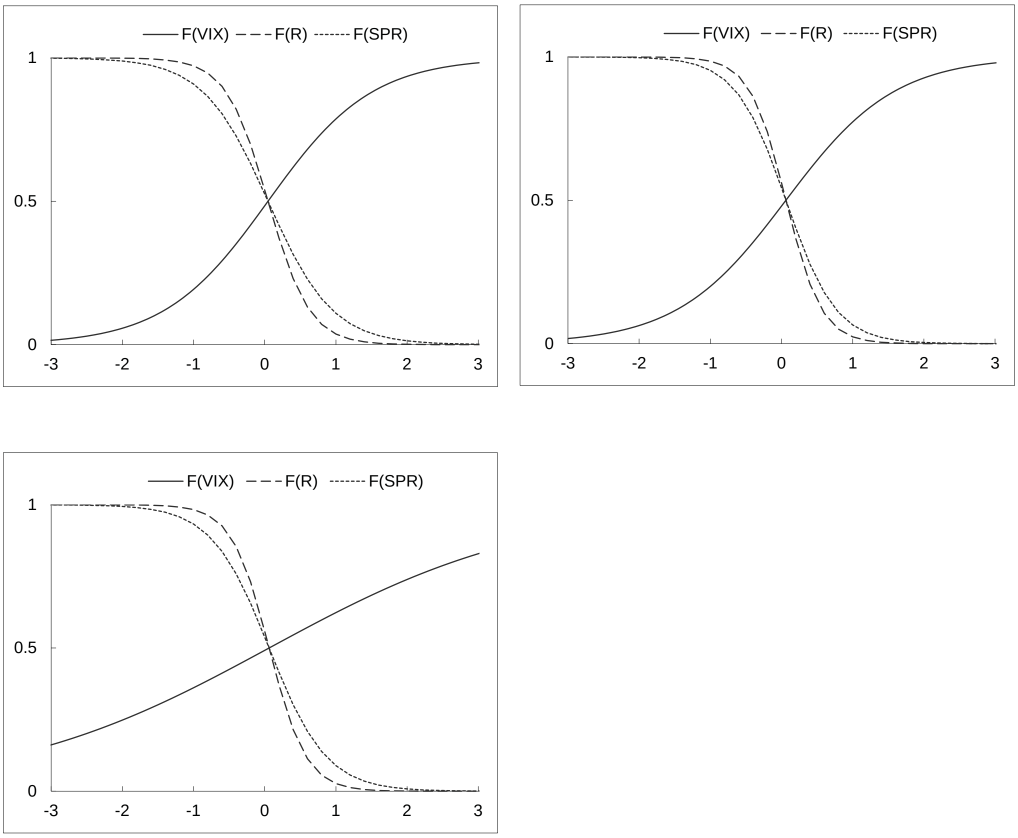
| $\phi$ | 0.380*** | 0.090 | 0.342** | 0.134 | 0.249*** | 0.080 |
| VIX | 1.370*** | 0.206 | 1.308*** | 0.099 | 0.537*** | 0.103 |
| R | -3.414*** | 1.018 | -3.968*** | 0.528 | -3.824*** | 0.097 |
| SPR | -2.201*** | 0.673 | -2.839*** | 0.610 | -2.476*** | 0.219 |
| c | 0.046 | 0.095 | 0.062 | 0.208 | -0.007 | 0.077 |
| Log-likelihood | -248.86 | | -250.95 | | -248.34 | |
| Linearity test | 12.3*** | | 24.44*** | | 16.55*** | |
| Additive nonlinearity test | 0.22 | | 0.73 | | 0.20 | |



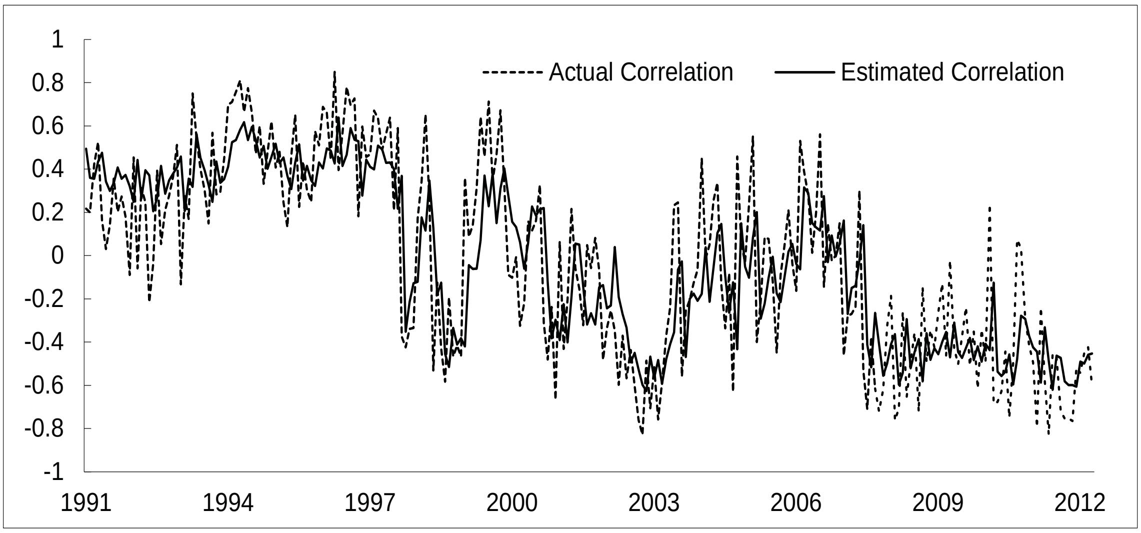
10. Plot the transition functions of each variable, holding the other variables constant at their mean values of zero
11. Correlation regime changes rather rapidly from the negative regime to the positive regime as short rates and yield spreads get larger
12. If short rate is lower (larger) than the average value by 1SD, the average correlation is less than $-0.30$ (more than 0.28) for the US
13. Stock-bond correlations tend to be positive when the economy is booming
14. VIX transition function also demonstrates flight-to-quality behavior
15. Estimated correlation fits the actual correlation quite well
16. Recent studies find long-run correlation trends in international financial markets
17. Instructive to examine whether we can modify Model 1 by introducing a time trend component

# Figure 1: Estimated transition function for Model 1






# Figure 2: Estimated stock-bond correlation for US (Model 1)



## Introduction of time trend component

1. Model 2: STR model with $s_{t-1} = (VIX_{t-1}, R_{t-1}, SPR_{t-1}, T_t)'$
2. Two-state model adequately captures all smooth transition regime-switching behavior in the data
3. Two distinct correlation regimes, with a negative average correlation for one regime and a positive average correlation for the other
4. AR term is significant at least at the 10% significance level for the US and GER
5. Time trend component coefficient estimates are significantly positive for all countries
6. There is a decreasing trend in stock-bond correlations
7. Rapid decrease between the late 1990s and the early 2000s, reaching an average of −0.42 by the end of sample period in May 2012

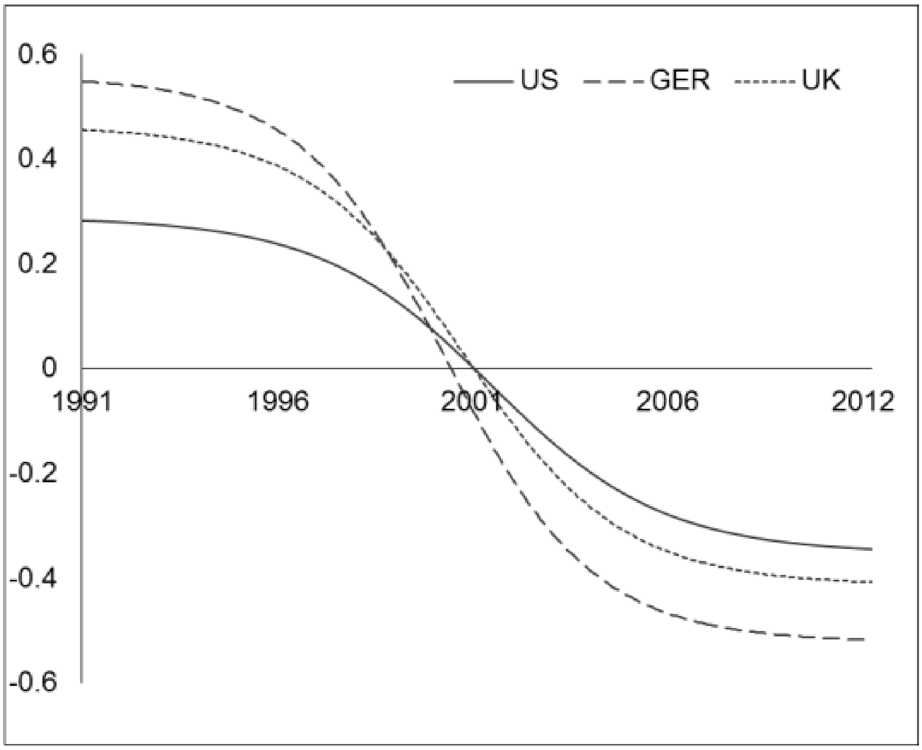


Table 2: Estimation results of the model with time trend component (Model 2)

| | US | | GER | | UK | |
|---|---|---|---|---|---|---|
| | Coef | St. err | Coef | St. err | Coef | St. err |
| $\rho_1$ | 0.297** | 0.140 | 0.630*** | 0.052 | 0.502*** | 0.117 |
| $\rho_2$ | -0.368*** | 0.099 | -0.580*** | 0.027 | -0.440 | 0.075 |
| $\phi$ | 0.346* | 0.192 | 0.140*** | 0.028 | 0.156 | 0.105 |
| VIX | 1.925*** | 0.616 | 1.142*** | 0.083 | 1.163*** | 0.354 |
| R | -0.576 | 0.461 | 1.323*** | 0.039 | 0.159 | 0.140 |
| SPR | -0.294 | 0.672 | 0.051 | 0.049 | -0.450*** | 0.161 |
| T | 2.571*** | 0.943 | 2.804*** | 0.010 | 2.725*** | 0.311 |
| c | 0.071 | 0.165 | -0.144*** | 0.054 | -0.065 | 0.158 |
| Log-likelihood | -248.23 | | -248.25 | | -247.29 | |
| Linearity test | 10.95*** | | 24.26*** | | 21.54*** | |
| Additive nonlinearity test | 1.28 | | 2.55 | | 0.09 | |

Figure 5: Estimated time trend component in the stock-bond correlation

(a) Model 2

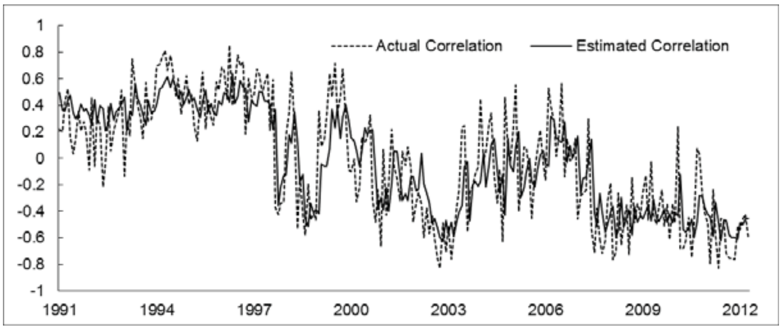




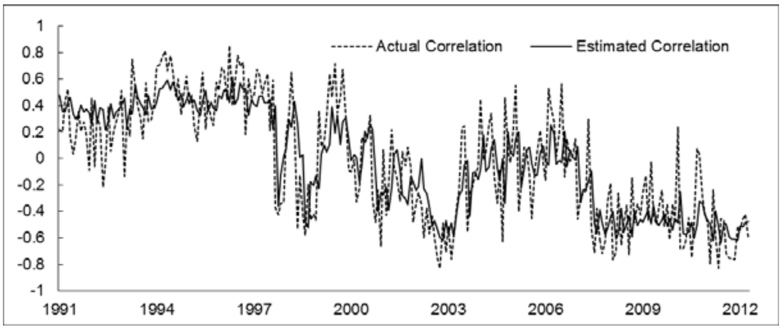

8. VIX remains an important factor in determining stock-bond correlations
9. Short rate and yield spread become less important in Model 2
10. Neither of Short rate and yield spread are significant for the US
11. Only one of them is significant for GER and the UK
12. Correlations estimated through Models 1 and 2 are similar to each other and do not differ much over the sample
13. AIC favors Model 2 for GER and the UK, while the SIC prefers Model 1 to Model 2 for all countries

Figure 2: Estimated stock-bond correlation for US

(a) Model 1




(b) Model 2






## Out-of-sample forecast evaluation

1. Conduct an out-of-sample forecast evaluation
   (a) Estimate both Models 1 and 2 using data from February 1991 to January 2001
   (b) Evaluate the terminal one-month-ahead forecast error based on the estimation results
   (c) Data are updated by one month
   (d) Terminal one-month-ahead forecast error is re-calculated from the updated sample
   (e) Repeat (c) and (d) until reaching one month before the end of the sample period
   (f) Calculate the root-mean-squared forecast errors (RMSE) and mean absolute error (MAE)
2. Model 2 performs better than Model 1 for GER
3. Model 1 exhibits better than Model 2 for other two countries

## Results with selected transition variables

1. Short rate and yield spread become less important determinants of stock-bond correlations if decreasing trends are accommodated
2. Possible to improve the model by excluding these variables
3. Model 3: STR model with $s_{t-1} = (VIX_{t-1}, T_t)'$
4. Estimation results are essentially same as those of Model 2
5. Stock-bond correlations in all countries exhibit clear decreasing trends, with a rapid decrease between 1999 and 2003
6. Estimated correlations are similar to those of other models
7. Model 3 is the best among the three models in terms of in-sample fit for all countries
8. Model 3 exhibits the best out-of-sample performance for all countries



Table 5: Estimation results of the parsimonious model (Model 3)

| | US | | GER | | UK | |
|---|---|---|---|---|---|---|
| | Coef | St. err | Coef | St. err | Coef | St. err |
| $\rho_1$ | 0.289*** | 0.001 | 0.459*** | 0.002 | 0.483*** | 0.185 |
| $\rho_2$ | -0.363*** | 0.002 | -0.570*** | 0.006 | -0.419** | 0.173 |
| $\phi$ | 0.359*** | 0.001 | 0.136*** | 0.005 | 0.173 | 0.192 |
| VIX | 1.983*** | 0.003 | 1.901*** | 0.009 | 1.373*** | 0.345 |
| T | 2.959*** | 0.003 | 3.315*** | 0.095 | 2.808*** | 0.675 |
| c | 0.068* | 0.041 | 0.005 | 0.067 | -0.106 | 0.192 |
| LLF | -248.27 | | -248.65 | | -247.51 | |
| Linearity test | 21.33*** | | 36.88*** | | 38.87*** | |
| Additive nonlinearity test | 1.25 | | 0.02 | | 0.61 | |

Figure 5: Estimated time trend component in the stock-bond correlation

(b) Model 3

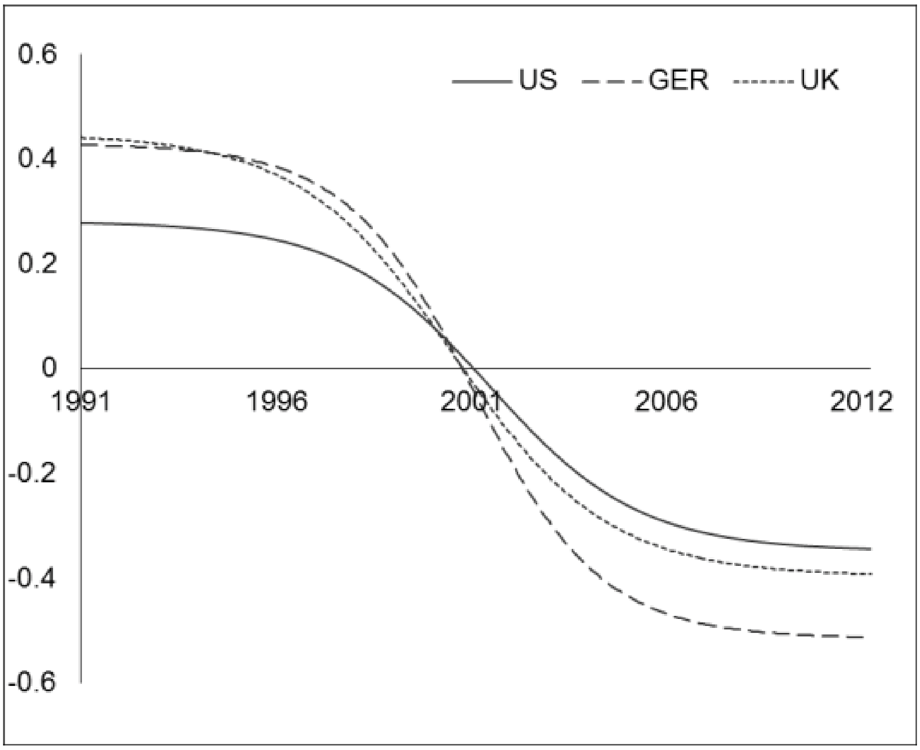
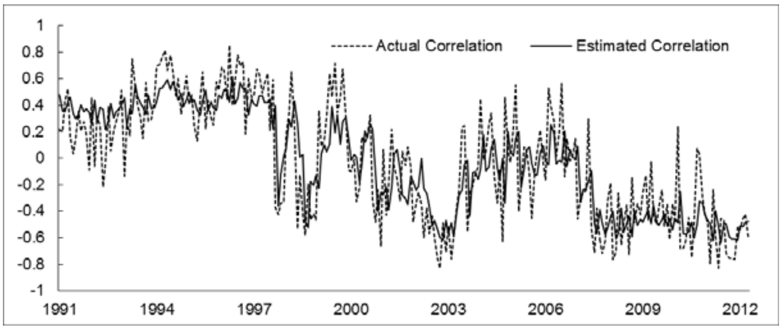





Figure 2: Estimated stock-bond correlation for US

(b) Model 2




(c) Model 3

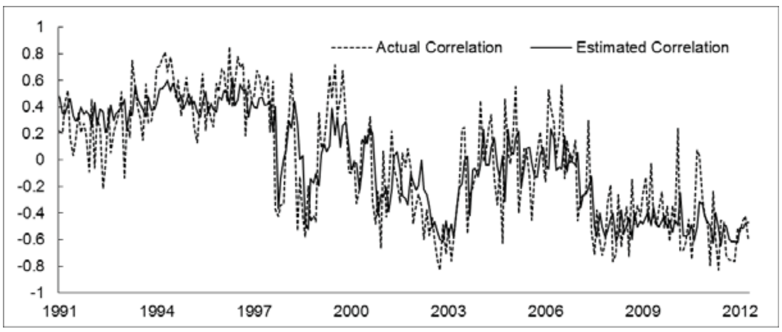


Table 3: Results of in-sample comparison

| | US | | GER | | UK | |
|---|---|---|---|---|---|---|
| | AIC | SIC | AIC | SIC | AIC | SIC |
| Model 1 | 511.72 | 536.54 | 515.90 | 540.71 | 510.68 | 535.50 |
| Model 2 | 512.46 | 540.82 | 512.51 | 540.87 | 510.58 | 538.95 |
| Model 3 | 508.54 | 529.81 | 509.30 | 530.58 | 507.01 | 528.28 |

Table 4: Results of out-of-sample comparison

| | US | | GER | | UK | |
|---|---|---|---|---|---|---|
| | RMSE | MAE | RMSE | MAE | RMSE | MAE |
| Model 1 | 0.201 | 0.155 | 0.322 | 0.257 | 0.259 | 0.212 |
| Model 2 | 0.203 | 0.161 | 0.297 | 0.231 | 0.274 | 0.221 |
| Model 3 | 0.174 | 0.136 | 0.296 | 0.231 | 0.241 | 0.199 |



## Interpretation of the results

1. Short rate and yield spread are not important factors in relation to stock-bond correlation regimes
2. Flight-to-quality behavior is not strongly related with economic conditions, but is associated with market uncertainty
3. Significant decreasing trends in stock-bond correlations
4. Flight-to-quality behavior has become stronger in more recent years
5. Many studies find an increasing trend in correlations in international equity markets as well as other financial markets
6. Diminish the effects of diversification in international financial markets
7. Investors need to make greater use of bond markets to control their risk exposure, producing decreasing trend in stock-bond correlations

## Conclusion

1. Examine the possible trend in stock-bond correlation for US, GER, UK
2. Find a significant decreasing trend in stock-bond correlations
3. Short rates and yield spreads become only marginally significant once we introduce the decreasing trend
4. STR model including the VIX and time trend as the transition variables dominates other models
5. Can be considered a consequence of the decreasing effects of diversification and more intensive flight-to-quality behavior



## Future topics

1. High frequency data
2. Model correlation as a latent variable
3. Asymmetric dependence
4. Source of long-run trends in international financial markets

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