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# 自己励起型強度モデルを用いた信用イベント発生インパクトの分析[1]

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**概 要**

本稿では，信用格付け変更の自己励起性および相互作用性について分析を行う．すなわち，格下げ（格上げ）の発生は格下げ発生頻度（格上げ発生頻度）を高めるかどうか，また，格下げ格上げの発生は互いの発生頻度にどのように影響するかを分析する．まず，発行体格付け変更データを用いて相互作用型イベント発生強度モデルの推定を行い，格付け変更に自己励起性および相互作用性が確認されるかどうか検証する．また，自己励起性の大きさを格付け変更時に観測される情報で説明することを試みる．

## 1 はじめに

本稿では，格下げ（格上げ）の発生が格下げ発生頻度（格上げ発生頻度）を高めるといった特徴（自己励起性），また，格下げ・格上げの発生がもう一方の発生頻度に影響を与えるといった特徴（相互作用性）に焦点をあてた分析を行う．具体的には，信用格付け変更データに対して自己励起性および相互作用性を表現し得る格下げ・格上げ発生強度モデルの推定を行い，その結果から格下げおよび格上げに自己励起性や相互作用性が確認されるか検証するとともに，自己励起性の大きさ（自己励起インパクト）と格付け変更発生時の情報との関係を説明することを試みる．

金融機関のリスク管理や金融商品の価格付けにおいては，デフォルトや信用格付け変更といった信用イベントの発生をモデル化し，信用リスクを評価することが必要になる．信用イベント発生の特徴として信用イベント発生のクラスターが生成されることが知られており，例えば日本で発生した発行体格付けの月次発生件数の時系列推移（図 1）をみても 1998 年から 2000 年, 2001 年から 2003 年，2008 年から 2010 年にかけて格下げがある程度まとまって発生している様子がわかる．信用イベント発生のクラスターは企業間の信用リスク依存関係の表れと解釈されており，複数の企業の信用リスク評価モデルを構成するにあたっては，そのような企業間の信用リスク依存関係をどのようにモデル化するかが重要になる．信用リスク評価の先行研究では，信用リスク依存関係をとらえるために信用イベント発生の伝播という特徴を取り入れたモデルが提案されている．信用イベント発生の伝

[1]本稿の内容は著者個人の見解であり，著者の所属機関の公式見解を示すものではない．本研究は文部科学省科学研究費補助金基盤研究 (A)「ファイナンス計量分析の新展開と金融市場」（課題番号 21243019）の支援を受けた．また、本研究は文部科学省グローバルＣＯＥプログラム「数学新展開の研究教育拠点」の支援を受けた．本稿の内容は Yamanaka et al. [10] の内容を基にして構成された．

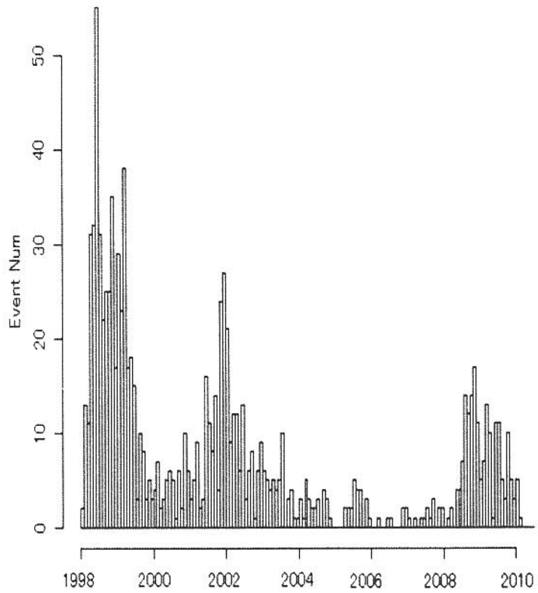


図 1: 日本における格下げ発生件数（月次）の推移．格付けは R&I の公表した発行体格付け．

播とは，ある企業の信用イベントの発生が，他の企業の信用イベント発生の可能性を高める，ということである．例えば，ボトムアップ・アプローチ[2] に基づく信用リスク評価の先行研究では，Jarrow and Yu [4] や Leung and Kwok [5] が個々の企業のデフォルト強度が他社のデフォルト発生によってジャンプするタイプのモデルを提案している．また，トップダウン・アプローチ[3] に基づく信用リスク評価の先行研究では，リスク伝播を表現するために自己励起性をもつ強度モデルが用いられてきた．自己励起性とは，イベント発生によってイベント発生強度を高まるという性質である．自己励起性強度過程による信用リスク評価を行った先行研究として Azizpour et al. [1], Errais et al. [2], Giesecke and Kim [3], Yamanaka et al. [9] などが挙げられ，いずれもあるタイプのイベントの発生によって同タイプのイベント発生強度が上方にジャンプするという構造をもつ自己励起型強度モデルを考えている．

さらに，単一の信用イベントだけでなく複数の異なるタイプの信用イベントを同時に考える場合には，自己励起性だけでなく相互作用性，すなわち異なるイベントの発生が互いに与える影響を表現するモデルが必要となってくる．異なるタイプの信用イベントが互いに与える影響を相互作用型強度モデルを用いて表現し，信用リスク評価や信用イベント発生データの分析を行った先行研究として Nakagawa[6] や中川 [7] が挙げられる．Nakagawa[6] や中川 [7] の相互作用型の強度モデルはあるタイプのイベント発生強度がそのタイプのイ

[2]個々の企業の信用イベント発生のモデル化を出発点とし，信用リスク依存関係を考慮して個々の信用イベント発生モデルから信用ポートフォリオの信用リスク評価モデルを構成するアプローチ.

[3]信用ポートフォリオ全体でのイベント発生モデルを構成し，ポートフォリオを構成する個々の企業の信用イベント発生モデルは必要に応じてポートフォリオ全体のモデルから構成するアプローチ.

ベント発生の影響だけでなく，別のタイプのイベント発生によってもジャンプするという形の相互作用性をもつモデルである．中川 [7] は相互作用性強度モデルを用いて，業種毎の格付け変更の自己励起性と業種間の格付け変更の相互作用性を確認している．また，格上げと格下げ間の相互作用性についての分析も行っている．

本稿では，信用格付け変更の自己励起性および相互作用性の分析を目的とした自己励起型，相互作用型の強度モデルを提案し，日本の信用格付け変更データに対するモデル推定を行う．具体的にはまず，中川 [7] の報告している格付け変更の自己励起性と格下げ・格上げ間の相互作用性を，業種に分けない形で，また中川 [7] とは異なる強度モデルを用いて分析する．そして，自己励起性インパクトの大きさの分析を行う．ここでは，自己励起インパクトの大きさが「変更元格付け」,「変更先格付け」,「格付け変更発生間隔」といった情報でどのように説明できるかを考察する．また，どの情報がインパクトの大きさを説明するうえで重要であるかという説明変数の選択を赤池情報量基準（AIC）に基づいて行う．

本稿の構成は以下になる．第 2 節では自己励起型および相互作用型強度モデルの定式化を行う．第 3 節では分析の対象となる格付け変更データの概要を述べる．第 4 節では，格付け変更データに自己励起性および相互作用性が確認されるか検証する．第 5 節では，自己励起性の大きさをいくつかの変数で説明することを試みる．第 6 節でまとめと今後の展望を述べる．

## 2　イベント発生強度モデル

本節では，信用イベント発生の自己励起性および相互作用性の分析を行うための相互作用型強度モデルの定式化を行う．$(\Omega,\ \mathcal{F},\ \mathbb{P},\ \{\mathcal{F}_t\})$ をフィルトレーション付き完備確率空間とする．ここで，$\mathbb{P}$ は実確率測度とし，$\{\mathcal{F}_t\}$ は右連続で完備なフィルトレーションとする．信用イベントのタイプを $\ell \in \{1,\ 2\}$ で表し， 特に $\ell = 1$ は格下げを $\ell = 2$ は格上げを表すことにする．各イベントタイプ $\ell$ に対して，マーク付き点過程 $\{(T_n^\ell, \zeta_n^\ell)\}_{n\in\mathbb{N}}$ を考える．$0 < T_1^\ell < T_2^\ell < \cdots$ は完全不到達な $\{\mathcal{F}_t\}$-停止時刻の列であり，イベントタイプ $\ell$ のイベント発生時刻列を表すとする．確率変数 $\zeta_n^\ell$ はイベント発生時刻 $T_n^\ell$ に付随する情報とする．イベント $\ell$ の発生件数を表す計数過程を $N_t^\ell = \sum_{n\ge1} \mathbf{1}_{\{T_n^\ell \le t\}}$ で表すことにする．また，異なるタイプのイベントは同時には発生しないことにする．

各 $N_t^\ell$ は強度過程 $\lambda_t^\ell$ をもつとする．すなわち，$\lambda_t^\ell$ は $\{\mathcal{F}_t\}$-発展的可測な非負の確率過程であり，$N_t^\ell - \int_0^t \lambda_s^\ell \mathrm{d}s$ が $\{\mathcal{F}_t\}$-局所マルチンゲールになるものである．本研究では $\lambda_t^\ell$ を次の相互作用型確率過程で与える:

$$\mathrm{d}\lambda_t^\ell \;=\; \kappa^\ell(c^\ell - \lambda_t^\ell)\mathrm{d}t + \mathrm{d}J_t^\ell,\ \lambda_0^\ell = c^\ell, \tag{1}$$

$$J_t^\ell \;=\; \sum_{n\ge1} f(\zeta_n^\ell)\mathbf{1}_{\{T_n^\ell \le t\}} + \sum_{n\ge1} g(\zeta_n^{\ell'})\mathbf{1}_{\{T_n^{\ell'} \le t\}}. \tag{2}$$

ここで，定数 $\kappa^\ell > 0$ と $c^\ell > 0$ はパラメタである．また，関数 $f(\cdot)$ は自己励起ジャンプの大きさを表し，$g(\cdot)$ は相互作用ジャンプの大きさを表す．すなわち，イベント発生強度 $\lambda_t^\ell$ はイベントタイプ $\ell$ のイベントが発生すると $f(\cdot)$ の大きさのジャンプをし，タイプ $\ell'$ のイベントが発生すると $g(\cdot)$ の大きさのジャンプをする．$g = 0$ の場合は相互作用によるジャ

ンプは発生しないため，相互作用過程は特に自己励起過程と呼ばれる．本稿では，$f(\cdot)$ を自己励起インパクトと呼ぶことにする．

第 4 節では，相互作用型強度モデルを用いて格付け変更に自己励起性あるいは相互作用性が存在するかどうかを検証する．すなわち，格付け変更データに対してモデルの推定を行い，$f \neq 0$ あるいは $g \neq 0$ かどうかを確認する．具体的には以下の二つのタイプのモデルを考える:

$$
\begin{aligned}
\text{モデル A} : J_t^\ell &= \sum_n \delta_1^\ell \mathbf{1}_{\{T_n^\ell \leq t\}} + \sum_n \delta_2^\ell \mathbf{1}_{\{T_n^{\ell'} \leq t\}}.\\
\text{モデル B} : J_t^\ell &= \sum_n \min(\delta_1^\ell \lambda_{T_n^\ell -}, \gamma^\ell) \mathbf{1}_{\{T_n^\ell \leq t\}}\\
&+ \sum_n \min(\delta_2^\ell \lambda_{T_n^{\ell'} -}, \gamma^\ell) \mathbf{1}_{\{T_n^{\ell'} \leq t\}}.
\end{aligned}
$$

ここで，定数 $\delta_1^\ell$，$\delta_2^\ell$，および $\gamma^\ell > 0$ はパラメタである．また、モデル B に対しては $\delta_1^\ell > -1$, $\delta_2^\ell > -1$ という条件を課すことにする．モデル A タイプのジャンプの構造をもつ強度モデルは，イベント $\ell$ が発生すると大きさ $\delta_1^\ell$ の自己励起ジャンプを起こし，イベント $\ell'$ が発生すると大きさが $\delta_2^\ell$ の相互作用ジャンプを起こすモデルである．モデル B のジャンプの大きさはイベント発生直前の強度の値に比例しており，上限 $\gamma^\ell$ をもつ．モデル A のジャンプは定数サイズであり，そのような簡潔な構造なためパラメータ推定が容易であるが，$\delta_1^\ell < 0$ あるいは $\delta_2^\ell < 0$ である場合に，ジャンプモデル A をもつ強度は負値を取り得る．従って，強度が非負であることがモデル A では保証されないという問題点がある．一方で，モデル B は比例定数に課された制約条件 $\delta_1^\ell > -1$ と $\delta_2^\ell > -1$ があり、この条件によって強度の値は非負に保たれる．

第 5 節では，自己励起インパクトの大きさといくつかの説明変数の関係を説明することを試みる．第 5 節では特に自己励起性に焦点を当てているため，自己励起ジャンプのみをもつモデルを考える．具体的には自己励起ジャンプの大きさが説明変数のアフィン関数で表現される次のモデルを考える:

$$
\text{モデル C} : J_t^\ell = \sum_n (a_0 + \sum_m a_m x_m(T_n^\ell)) \mathbf{1}_{\{T_n^\ell \leq t\}}.
$$

ここで，$a_0$ は定数項，$\{x_m\}$ は説明変数であり，$\{a_m\}$ は係数である．

表 1 は本稿で用いるモデルの特徴の一覧である．

表 1: 各ジャンプモデルの特徴

| モデル | 自己励起ジャンプ | 相互作用ジャンプ | 用途 |
|---|---|---|---|
| A | 定数 | 定数 | 自己励起性・相互作用性の存在確認 |
| B | 強度比例 (制約あり) | 強度比例 (制約あり) | 自己励起性・相互作用性の存在確認 |
| C | 説明変数の関数 | - | 自己励起ジャンプの説明 |

## 3 データ

分析に用いるデータは 1998 年 4 月 1 日から 2009 年 3 月 31 日の間に R&I の公表した日本企業の発行体格付け変更データである．各格付け変更データは，格付け変更日，発行体名，業種，イベントのタイプ，変更前格付け，変更後格付けからなる．また，データには保険会社の保険金支払い能力に関する格付けも含まれている．このデータ期間に格下げは 965 件，格上げは 481 件発生している．サンプルデータには信用力が高い順に AAA, AA+, $\cdots$, C− の 25 段階の格付けがあり，本稿では順に $1, 2, \cdots, 25$ で表すことにする．また，強度による分析を行うために格付け変更日を連続時間に直した．具体的には，休日を除いた上で 2004/4/1, 2005/4/1,$\cdots$ が $t = 0, 1, \cdots$ に対応するように時刻の変換した．さらに，同日に起こった複数の格付け変更に対しては，一様乱数を用いて変更時刻をランダムにずらすことで，異なる発生時刻を割り当てた[4]．

図 2, 3 および 4 はそれぞれ "変更前格付け", "格付け変更幅" および "格付け変更発生時刻間隔" の分布である．図 2 から，格付け変更の多くは格付け 4 から 11 の間の格付けからの変更であることがわかる．図 3 から多くの格付け変更が 1 段階あるいは 2 段階の格付け変更であることがわかる．図 4 から格付け変更の或る程度が同日のうちに起こっていることがわかる．また，格付け変更はその多くが直前の格付け変更から一週間以内（5 営業日以内）に発生していることがわかる．

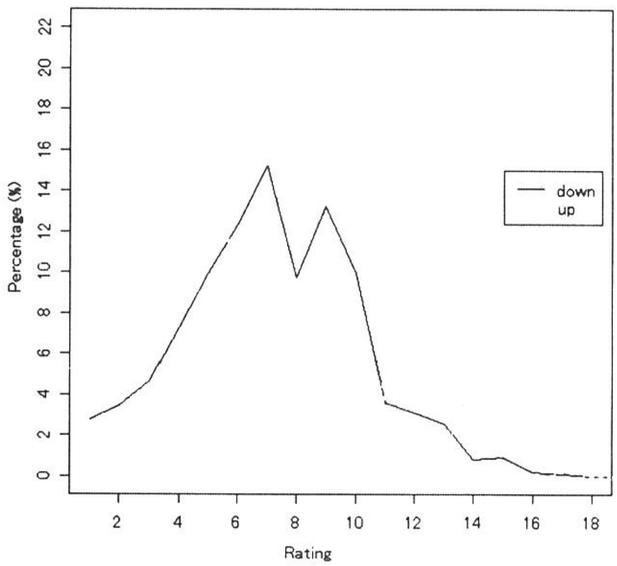


図 2: 変更前の格付けの分布．サンプルの期間は 1998/4/1 から 2009/3/31.

[4]イベントが複数起こる日が十分に数多くあるため，このようなデータ加工の分析結果に与える影響は小さいと考える．

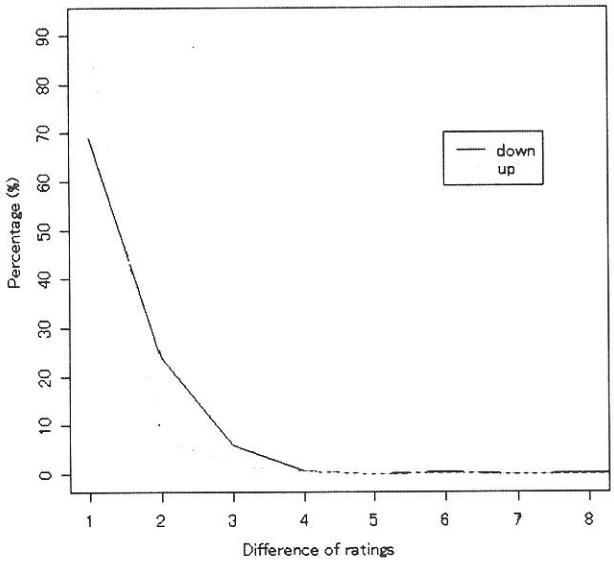


図 3: 格付け変更幅の分布. サンプルの期間は 1998/4/1 から 2009/3/31.

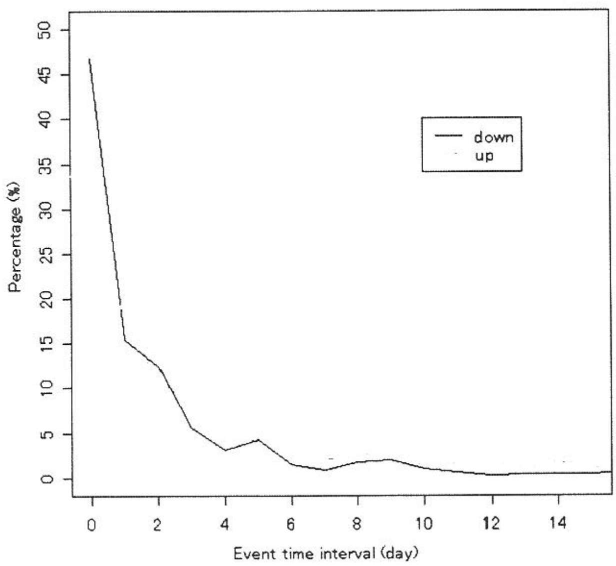


図 4: 格付け変更の発生間隔. サンプルの期間は 1998/4/1 から 2009/3/31.

## 4　自己励起性および相互作用性の存在

本節では格下げや格上げに自己励起性や相互依存性といった特徴がみられるかどうかを分析する．すなわち，格下げや格上げの発生時に格下げ，格上げ強度がジャンプするかどうかの検証を行う．

自己励起性や相互依存性を確認するために，ジャンプの形がモデルAおよびモデルBであるような強度モデルの推定を行う．推定された自己励起ジャンプの大きさが有意に $\delta_1^\ell \neq 0$ であれば，自己励起性が存在するとする．同様に，相互依存ジャンプの大きさが有意に $\delta_2^\ell \neq 0$ と推定されれば，相互依存性が存在するとする．モデルの推定は最尤法に基づいて行う．イベント $\ell$ の発生強度の対数尤度関数は次に表されることが知られている:

$$\sum_{n=1}^{N} \log \lambda^\ell_{T_n^\ell -} - \int_0^H \lambda_s^\ell \mathrm{d}s.$$

ここで，$\lambda_{t-}^\ell := \lim_{s\uparrow t} \lambda_s^\ell$ である．尤度関数を最大化するパラメタの探索には，統計解析ソフトウェアRの最適化関数`optim`を用いた．推定を容易にするために，モデルBのパラメタ $\gamma^\ell$ は，格下げの場合 $\gamma^1 = 100, 125, 150, 175, 200$ の中から，格上げについては $\gamma^2 = 5.0, 7.5, 10.0, 12.5, 15.0$ の値の中から選ばれるように限定した．推定結果の解釈については，推定されたパラメタの絶対値が標準推定誤差の2倍 (95%有意水準に対応) より大きい場合に，自己励起性や相互依存性が有意に確認されたとすることにする．

表2はモデルAおよびBの推定結果である．推定された格下げ強度のジャンプの大きさは有意に $\delta_1^1 \neq 0$ と $\delta_2^1 \neq 0$ である．このことより，格下げ発生強度には自己励起性と格上げからの影響（片側の相互作用性）をもつことが示唆される．特に自己励起ジャンプは有意に $\delta_1^1 > 0$ であることから格下げ発生の可能性は格下げ発生により高まり，また，相互作用ジャンプは有意に $\delta_2^1 < 0$ であるため，格下げ発生の可能性は格上げ発生により低くなる，といえる．格上げ強度については，推定された自己励起ジャンプの大きさが有意に $\delta_1^2 \neq 0$ であり，自己励起性の存在が示唆された．特に，有意に $\delta_1^2 > 0$ であることから，格上げの発生によって格上げ発生確率が高まることが示唆される．一方で，格上げ強度の相互依存性については，ジャンプの大きさがモデルAとBともに $\delta_2^2 < 0$ であるものの，モデルAについては $\delta_2^2 < 0$ であることは有意ではなかった．従って，格上げ強度に対する格下げからの相互作用性は，はっきりとは確認されなかった．

## 5　自己励起インパクトの説明

### 5.1　自己励起インパクトと説明変数の関係

本節では，自己励起インパクト，すなわち自己励起型強度モデルのジャンプの大きさをイベント発生時に観測されるいくつかの情報で説明することを試みる．具体的にはモデルCに対して以下の情報を説明変数の候補として用意し，パラメタ推定を行うことにする:

- $x_1(T_n^\ell) \in \{1, 2, \cdots, K\}$: 時刻 $T_n^\ell$ における格付け変更後の格付け（変更後格付），
- $x_2(T_n^\ell) \in \{1, 2, \cdots, K\}$: 時刻 $T_n^\ell$ における格付け変更前の格付け（変更前格付），

表 2: モデル A と B の推定結果．括弧内は標準推定誤差．$\gamma^\ell$ の推定値は $\gamma^1 = 125$ と $\gamma^2 = 10.0$ である．

| | モデル | $\kappa^\ell$ | $c^\ell$ | $\delta_1^\ell$ | $\delta_2^\ell$ |
|---|---|---|---|---|---|
| | A | 234.43 | 33.19 | 150.36 | −9.20 |
| 格下げ | | (42.33) | (3.09) | (22.64) | (2.10) |
| $\ell = 1$ | B | 170.66 | 33.59 | 2.32 | −0.38 |
| | | (9.89) | (2.75) | (0.39) | (0.09) |
| | A | 7.24 | 5.92 | 6.37 | −0.042 |
| 格上げ | | (2.01) | (2.91) | (1.72) | (0.095) |
| $\ell = 2$ | B | 11.75 | 13.14 | 0.28 | −0.021 |
| | | (0.54) | (0.81) | (0.026) | (0.009) |

- $x_3(T_n^\ell) = x_1(T_n^\ell) - x_2(T_n^\ell)$: 時刻 $T_n^\ell$ における格付け変更幅（格付け変更幅），
- $x_4(T_n^\ell) = T_n^\ell - T_{n-1}^\ell$: $n-1$ 番目の格付け変更と $n$ 番目の格付け変更発生の時間間隔（格付け変更間隔）．

$x_1(T_n^\ell)$ と $x_2(T_n^\ell)$ と $x_3(T_n^\ell)$ を同時に考えると情報が重なるため，$x_1(T_n^\ell)$ を除いた $x_2(T_n^\ell)$, $x_3(T_n^\ell)$ および $x_4(T_n^\ell)$ を説明変数として採用することにする．ここで，$x_3$ は現在の格付けと前回の格付けの絶対差ではなく，単なる差であることに注意する．

表 3 がモデル C の推定結果である．格下げに関しては $a_2 < 0$, $a_3 > 0$ と $a_4 < 0$ であることが有意に推定されたことから，, 以下の 3 つの場合について自己励起インパクトが大きくなることが示唆される:

- 格下げ前の格付けの信用力が高い．
- 格付け変更の幅が広い．
- 前回の格下げ発生から今回の格下げ発生までの時間間隔が短い．

一方で，格上げについては，各係数の推定誤差が大きいため，有意な結果を得ることはできなかった．

表 3: 自己励起インパクトの推定結果．括弧内は標準推定誤差．

| | $a_0$ | $a_2$ | $a_3$ | $a_4$ |
|---|---|---|---|---|
| 格下げ | 193.53 | −11.56 | 34.57 | −380.18 |
| | (36.96) | (3.37) | (14.97) | (137.64) |
| 格上げ | 9.43 | −0.09 | 1.81 | −7.38 |
| | (5.35) | (0.53) | (2.48) | (12.52) |

### 5.2　説明変数の選択

本小節では，前述した自己励起インパクトの説明変数の候補のうち，どの情報がより説明力をもつかについて検討する．具体的には，モデル C に採用する説明変数の組み合わせとして，AIC が小さい説明変数の組み合わせをを選択することにする．変数の組み合わせを比較する際には，AIC の差が 1 を超える場合に AIC の差が有意であるとする ([8])．説明変数は，先に述べた現在の格付け $x_1(T_n^\ell)$, 変更前の格付け $x_2(T_n^\ell)$, 格付け変更幅 $x_3(T_n^\ell)$ と格付け変更発生間隔 $x_4(T_n^\ell)$ の 4 つとする．

表 4 が格下げの自己励起インパクトに関する説明変数の選択結果である．表 4 から以下の結果が示唆される:

- “格下げ発生間隔”は “変更後格付け” と “変更前格付け”が情報として存在する場合にあまり重要ではない．
- “変更前格付け”は “変更後格付け”より説明力が高い．
- “格付け変更幅” は “変更後格付け”や “変更後格付け”よりも重要度が低い．

表 4: 格下げに関する自己励起インパクトの説明変数の選択．変数の組み合わせに採用している情報には 1 というフラグを立ててある．説明変数の組み合わせの並びは AIC の昇順である．

| 定数 | $x_1$ | $x_2$ | $x_3$ | $x_4$ | パラメタ数 | 対数尤度 | AIC |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | | | 5 | 3865.09 | −7720.2 |
| 1 | 1 | 1 | | 1 | 6 | 3865.88 | −7719.8 |
| 1 | | 1 | | 1 | 5 | 3862.66 | −7715.3 |
| 1 | | 1 | | | 4 | 3861.02 | −7714.0 |
| 1 | 1 | | | 1 | 5 | 3860.63 | −7711.3 |
| 1 | 1 | | | | 4 | 3858.74 | −7709.5 |
| 1 | | | 1 | 1 | 5 | 3858.91 | −7707.8 |
| 1 | | | | 1 | 4 | 3857.42 | −7706.8 |
| 1 | | | 1 | | 4 | 3857.41 | −7706.8 |
| 1 | | | | | 3 | 3855.45 | −7704.9 |

表 5 は格上げに関する自己励起インパクトの説明変数選択結果である．格下げの結果とは異なり，説明変数を増やすことが尤度関数の増加に効果的に反映されず，説明変数の少ないモデルが選択される結果となった．この結果は，今回の格上げサンプルデータに対して説明変数が有効でなかったことを示唆する．

## 6　まとめ

格付け変更の自己励起性と相互作用性について分析を行った．まず，二つのタイプの相互作用性強度モデルを用いて，格付け変更に自己励起性と相互作用性が確認されるかにつ

表 5: 格上げに関する自己励起インパクトの説明変数の選択．変数の組み合わせに採用している情報には 1 というフラグを立ててある．説明変数の組み合わせの並びは AIC の昇順である．

| 定数 | $x_1$ | $x_2$ | $x_3$ | $x_4$ | パラメタ数 | 対数尤度 | AIC |
|---|---|---|---|---|---|---|---|
| 1 | | | | | 3 | 1503.2 | −3000.5 |
| 1 | | | 1 | | 4 | 1503.5 | −2999.0 |
| 1 | | | | 1 | 4 | 1503.3 | −2998.5 |
| 1 | | 1 | | | 4 | 1503.3 | −2998.5 |
| 1 | 1 | | | | 4 | 1503.2 | −2998.5 |
| 1 | | | 1 | 1 | 5 | 1503.5 | −2997.0 |
| 1 | 1 | 1 | | | 5 | 1503.5 | −2997.0 |
| 1 | | 1 | | 1 | 5 | 1503.3 | −2996.6 |
| 1 | 1 | | | 1 | 5 | 1503.3 | −2996.5 |
| 1 | 1 | 1 | | 1 | 6 | 1503.5 | −2995.0 |

いて検証した．強度モデルのジャンプの大きさの推定から，格下げについては自己励起性と格上げからの影響（片側の相互作用性）が存在することを示唆する結果を得た．また，格上げについても自己励起性が確認された．続いて，自己励起インパクトいくつかの変数を用いて説明することを試みた．そこでは，ジャンプの形が説明変数のアフィン関数で表される自己励起強度モデルを考えた．説明変数の選択は，AIC に基づいて行った．格下げについては自己励起インパクトの大きさと説明変数の関係を有意に説明する結果を得たが，格上げについては有意な結果は得られなかった．

今回は推定の容易さを考慮して，説明変数として格付けと格付け変更発生間隔に関する情報を利用した．しかし，その他の情報を説明変数と利用することも可能であり，より多くの説明変数を考慮した分析は今後の課題である．また，本稿のモデル C は自己励起ジャンプの大きさと他の変数の関係を表現できており，ジャンプの大きさが定数であったり他の変数と独立であった先行研究のモデル（Azizpour et al. [1], Errais et al. [2], Giesecke and Kim [3], Yamanaka et al. [9] など）よりもリスク伝播を柔軟にとらえることができる．本稿の強度モデルによってリスク伝播，リスク依存関係を捉えた信用リスク評価も今後の課題である．

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# Derivative Pricing under Asymmetric and Imperfect Collateralization and CVA *

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**Abstract**

The importance of collateralization through the change of funding cost is now well recognized among practitioners. In this article, we have extended the previous studies of collateralized derivative pricing to more generic situation, that is asymmetric and imperfect collateralization as well as the associated CVA. We have presented approximate expressions for various cases using Gateaux derivative which allow straightforward numerical analysis. Numerical examples for CCS (cross currency swap) and IRS (interest rate swap) with asymmetric collateralization were also provided. They clearly show the practical relevance of sophisticated collateral management for financial firms. The valuation and the associated issue of collateral cost under the one-way CSA (or unilateral collateralization), which is common when SSA (sovereign, supranational and agency) entities are involved, have been also studied. We have also discussed some generic implications of asymmetric collateralization for netting and resolution of information.

**Keywords :** swap, collateral, derivatives, Libor, currency, OIS, EONIA, Fed-Fund, CCS, basis, risk management, HJM, FX option, CSA, CVA, term structure, SSA, one-way CSA

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# 1 Introduction

In the last decade, collateralization has experienced dramatic increase in the derivative market. According to the ISDA survey [11], the percentage of trade volume subject to collateral agreements in the OTC (over-the-counter) market has now become 70%, which was merely 30% in 2003. If we focus on large broker-dealers and the fixed income market, the coverage goes up even higher to 84%. Stringent collateral management is also a crucial issue for successful installation of CCP (central clearing parties).

Despite its long history in the financial market as well as its critical role in the risk management, it is only after the explosion of Libor-OIS spread following the collapse of Lehman Brothers that the effects of collateralization on derivative pricing have started to gather strong attention among practitioners. In most of the existing literatures, collateral cost has been neglected, and only its reduction of counterparty exposure have been considered. The work of Johannes & Sundaresan (2007) [12] was the first focusing on the cost of collateral, which studied its effects on swap rates based on empirical analysis. As a more recent work, Piterbarg (2010) [13] discussed the general option pricing using the similar formula to take the collateral cost into account.

In a series of works of Fujii, Shimada & Takahashi (2009) [7, 8] and Fujii & Takahashi (2010,2011) [9, 10], modeling of interest rate term structures under collateralization has been studied, where cash collateral is assumed to be posted continuously and hence the remaining counterparty credit risk is negligibly small. In these works, it was found that there exists a direct link between the cost of collateral and CCS (cross currency swap) spreads. In fact, one cannot neglect the cost of collateral to make the whole system consistent with CCS markets, or equivalently with FX forwards. Making use of this relation, we have also shown the significance of a "cheapest-to-deliver" (CTD) option implicitly embedded in a collateral agreement in Fujii & Takahashi (2011) [10].

The previous works have assumed bilateral and symmetric collateralization, where the two parties post the same currency or choose the optimal one from the same set of eligible currencies. Although symmetric collateral agreement is widely used, asymmetric situation can also arise in the actual market. If there is significant difference in credit qualities between two parties, the relevant CSA (credit support annex, specifying all the details of collateral agreements) may specify asymmetric collateral treatments, such as unilateral collateralization and asymmetric collateral thresholds. Especially, when SSA(sovereign, supranational and agency) clients are involved, one-way CSA is quite common: SSA entities refuse to post collateral but require it from the counterpart financial firms. One-way CSA is now becoming a hot issue among practitioners [14]. Since the financial firm needs to enter two-way CSA (or bilateral collateralization) to hedge the position in financial market, there appears a significant cash-flow mismatch. In addition, as we will see later, the financial firm may suffer from the significant loss of mark-to-market value due to the rising cost of collateral.

Asymmetric collateralization, even if the details specified in CSA are symmetric, may also arise effectively due to the different level of sophistication of collateral management between the two parties. For example, one party can only post single currency due to the lack of easy access to foreign currency pools or flexible operational system while the other chooses the cheapest currency each time it posts collateral. It should be also important to understand the change of CVA (credit value adjustment) under collateralization.

Although, it is reasonable in normal situations to assume most of the credit exposure is eliminated by collateralization for standard products, such as interest rate swaps, preparing for credit exposure arising from the deviation from the perfect collateral coverage should be very important for the risk management, particularly for complex path-dependent contracts, for which it is unlikely to achieve complete price agreements between the two parties.

This work has extended the previous research to the more generic situations, that is asymmetric and imperfect collateralization. The formula for the associated CVA is also derived. We have examined a generic framework which allows asymmetry in a collateral agreement and also imperfect collateralization, and then shown that the resultant pricing formula is quite similar to the one appearing in the work of Duffie & Huang (1996) [3]. Although the exact solution is difficult to obtain, Gateaux derivative allows us to get approximate pricing formula for all the cases in the unified way. In order to see the quantitative impacts, we have studied IRS (interest rate swap) and CCS with an asymmetric collateral agreement. We have shown the practical significance for both cases, which clearly shows the relevance of sophisticated collateral management for all the financial firms. Those carrying out optimal collateral strategy can enjoy significant funding benefit, while the others incapable of doing so will have to pay unnecessary expensive cost. We also found the importance of cost of collateral for the evaluation of CVA. The present value of future credit exposure can be meaningfully modified due to the change of effective discounting rate, and can be also affected by the possible dependency between the collateral coverage ratio and the counter party exposure. There also appear a new contribution called CCA (collateral cost adjustment) that purely represents the adjustment of collateral cost due to the deviation from the perfect collateralization.

After the collapse of Lehman Brothers, investors have been suffering from the loss of transparency of prices provided by broker-dealers, each of them quotes quite different bids and offers. This is mainly because the financial firms started to pay more attention to counter party credit risk and also because there was no consensus for the proper method of discounting of future cash flows for secured contracts with collateral agreements. However, the situation is now changing. Recently, SwapClear of LCH.Clearnet group, which is one of the largest clearing house in the world, started to use OIS (overnight index swap) curve to discount the future cash flows of swaps. This is one of the examples that the market benchmark quotes for the standardized products are converging to the perfectly collateralized ones with standard symmetric CSA. We also think that this should be the only possible way to achieve enough price transparency, since otherwise we need the portfolio and counterparty specific adjustment. Our formulation is based on the above understanding and derives CCA and CVA as a deviation from the collateralized benchmark price, which should be useful for practitioners who are required clear explanation for each additional charge to their clients.

We have also discussed some interesting implications for financial firm's behavior under (almost) perfect collateralization. One observes that the strong incentives for advanced financial firms to exploit funding benefit may reduce overall netting opportunities in the market, which can be a worrisome issue for the reduction of the systemic risk in the market.

## 2 Generic Formulation

In this section, we consider the generic pricing formula. As an extension from the previous works, we allow asymmetric and/or imperfect collateralization with bilateral default risk. We basically follow the setup in Duffie & Huang (1996) [3] and extend it so that we can deal with cost of collateral explicitly. The approximate pricing formulas that allow simple analytic treatment are derived by Gateaux derivatives.

### 2.1 Fundamental Pricing Formula

#### 2.1.1 Setup

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$, where $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ is sub-$\sigma$-algebra of $\mathcal{F}$ satisfying the *usual* conditions. Here, $Q$ is the spot martingale measure, where the money market account is being used as the numeraire. We consider two counterparties, which are denoted by party 1 and party 2. We model the stochastic default time of party $i$ ($i \in \{1,2\}$) as an $\mathbb{F}$-stopping time $\tau^i \in [0, \infty]$, which are assumed to be totally inaccessible. We introduce, for each $i$, the default indicator function, $H_t^i = \mathbf{1}_{\{\tau^i \leq t\}}$, a stochastic process that is equal to one if party $i$ has defaulted, and zero otherwise. The default time of any financial contract between the two parties is defined as $\tau = \tau^1 \wedge \tau^2$, the minimum of $\tau^1$ and $\tau^2$. The corresponding default indicator function of the contract is denoted by $H_t = \mathbf{1}_{\{\tau \leq t\}}$. The Doob-Meyer theorem implies the existence of the unique decomposition as $H^i = A^i + M^i$, where $A^i$ is a predictable and right-continuous (it is continuous indeed, since we assume total inaccessibility of default time), increasing process with $A_0^i = 0$, and $M^i$ is a $Q$-martingale. In the following, we also assume the absolute continuity of $A^i$ and the existence of progressively measurable non-negative process $h^i$, usually called the hazard rate of counterparty $i$, such that

$$A_t^i = \int_0^t h_s^i \mathbf{1}_{\{\tau^i > s\}} ds, \qquad t \geq 0 \ . \tag{2.1}$$

For simplicity we also assume that there is no simultaneous default with positive probability and hence the hazard rate for $H_t$ is given by $h_t = h_t^1 + h_t^2$ on the set of $\{\tau > t\}$.

We assume collateralization by cash which works in the following way: *if the party $i$ ($\in \{1,2\}$) has negative mark-to-market, it has to post the cash collateral* [1] *to the counter party $j$ ($\neq i$), where the coverage ratio of the exposure is denoted by $\delta_t^i \in \mathbb{R}_+$. We assume the margin call and settlement occur instantly. Party $j$ is then a collateral receiver and has to pay collateral rate $c_t^i$ on the posted amount of collateral, which is $\delta_t^i \times$ (|mark-to-market|), to the party $i$. This is done continuously until the end of the contract. A common practice in the market is to set $c_t^i$ as the time-$t$ value of overnight (ON) rate of the collateral currency used by the party $i$.* We emphasize that it is crucially important to distinguish the ON rate $c^i$ from the theoretical risk-free rate of the same currency $r^i$, where both of them are progressively measurable. The distinction is necessarily for the unified treatment of different collaterals and for the consistency with cross currency basis spreads, or equivalently FX forwards in the market (See, Sec. 6.4 and Ref. [10] for details.).

[1] According to the ISDA survey [11], more than 80% of collateral being used is cash. If there is a liquid repo or security-lending market, we may also carry out similar formulation with proper adjustments of its funding cost.

We consider the assumption of continuous collateralization is a reasonable proxy of the current market where daily (even intra-day) margin call is becoming popular. We are mainly interested in well-collateralized situation where $\delta_t^i \simeq 1$, however, we do also include the under- as well as over-collateralized cases, in which we have $\delta_t^i < 1$ and $\delta_t^i > 1$, respectively. Although it may look slightly odd to include the $\delta_t^i \neq 1$ case under the continuous assumption at first sight, we think that allowing under- and over-collateralization makes the model more realistic considering the possible price dispute between the relevant parties, which is particularly the case for exotic derivatives. Most of the long dated exotics, such as PRDC and CMS-related products, contain path-dependent knock-out or early redemption triggers, which makes the sizable price disagreements between the two parties almost inevitable. Because of the model uncertainty, the price reconciliation is usually done in ad-hoc way, say taking an average of each party's quote. As a result, even after the each margin settlement, there always remains sizable discrepancy between the collateral value and the model implied fair value of the portfolio. Therefore, even in the presence of timely margining, the inclusion of generic collateral coverage ration taking value bigger or smaller than 1 should be important for portfolios containing exotics.

Under the assumption, the remaining credit exposure of the party $i$ to the party $j$ at time $t$ is given by

$$\max(1-\delta_t^j,0)\max(V_t^i,0)+\max(\delta_t^i-1,0)\max(-V_t^i,0) \ ,$$

where $V_t^i$ denotes the mark-to-market value of the contract from the view point of party $i$. The second term corresponds to the over-collateralization, where the party $i$ can only recover the fraction of overly posted collateral when party $j$ defaults. We denote the recovery rate of the party $j$, when it defaults at time $t$, by the progressively measurable process $R_t^j \in [0,1]$. Thus, the recovery value that the party $i$ receives can be written as

$$R_t^j\Big(\max(1-\delta_t^j,0)\max(V_t^i,0)+\max(\delta_t^i-1,0)\max(-V_t^i,0)\Big) \ . \tag{2.2}$$

As for notations, we will use a bracket "( )" when we specify type of currency, such as $r_t^{(i)}$ and $c_t^{(i)}$, the risk-free and the collateral rates of currency $(i)$, in order to distinguish it from that of counter party. We also denote a spot FX at time $t$ by $f_{xt}^{(i,j)}$ that is the price of a unit amount of currency $(j)$ in terms of currency $(i)$. We assume all the technical conditions for integrability are satisfied throughout this paper.

### 2.1.2 Pricing Formula

We consider the ex-dividend price at time $t$ of a generic financial contract made between the party 1 and 2, whose maturity is set as $T$ $(> t)$. We consider the valuation from the view point of party 1, and define the cumulative dividend $D_t$ that is the total receipt from party 2 subtracted by the total payment from party 1. We denote the contract value as $S_t$ and define $S_t = 0$ for $\tau \leq t$. See Ref.[3] for the technical details about the regularity conditions which guarantee the existence and uniqueness of $S_t$. Under these assumptions and the setup give in Sec.2.1.1, one obtains

$$S_t = \beta_t E^Q \left[ \int_{]t,T]} \beta_u^{-1} \mathbf{1}_{\{\tau>u\}} \left\{ dD_u + \left( y_u^1 \delta_u^1 \mathbf{1}_{\{S_u<0\}} + y_u^2 \delta_u^2 \mathbf{1}_{\{S_u\geq 0\}} \right) S_u du \right\} \right.$$
$$\left. + \int_{]t,T]} \beta_u^{-1} \mathbf{1}_{\{\tau\geq u\}} \left( Z^1(u, S_{u-}) dH_u^1 + Z^2(u, S_{u-}) dH_u^2 \right) \middle| \mathcal{F}_t \right], \quad (2.3)$$

on the set of $\{\tau > t\}$. Here, $y^i = r^i - c^i$ denotes a spread between the risk-free and collateral rates of the currency used by party $i$, which represents the instantaneous return from the collateral being posted, i.e. it earns $r^i$ but subtracted by $c^i$ as the payment to the collateral payer. $\beta_t = \exp\left(\int_0^t r_u du\right)$ is a money market account for the currency on which $S_t$ is defined. $Z^i$ is the recovery payment from the view point of the party 1 at the time of default of party $i$ ($\in \{1, 2\}$):

$$Z^1(t, v) = \left(1 - (1 - R_t^1)(1 - \delta_t^1)^+\right) v \mathbf{1}_{\{v<0\}} + \left(1 + (1 - R_t^1)(\delta_t^2 - 1)^+\right) v \mathbf{1}_{\{v\geq 0\}} \quad (2.4)$$
$$Z^2(t, v) = \left(1 - (1 - R_t^2)(1 - \delta_t^2)^+\right) v \mathbf{1}_{\{v\geq 0\}} + \left(1 + (1 - R_t^2)(\delta_t^1 - 1)^+\right) v \mathbf{1}_{\{v<0\}}, \quad (2.5)$$

where $X^+$ denotes $\max(X, 0)$. Note that the above definition is consistent with the setup in Sec.2.1.1. The surviving party loses money if the received collateral from the defaulted party is not enough or if the posted collateral to the defaulted party exceeds the fair contract value.

Even if we explicitly take the cost of collateral into account, it is possible to prove the following proposition about the pre-default value of the contract in completely parallel fashion with the one given in [3]:

**Proposition 1** *Suppose a generic financial contract between the party 1 and 2, of which cumulative dividend at time $t$ is denoted by $D_t$ from the view point of the party 1. Assume that the contract is continuously collateralized by cash where the coverage ratio of the party $i$ ($\in \{1, 2\}$)'s exposure is denoted by $\delta_t^i \in \mathbb{R}_+$. The collateral receiver has to pay collateral rate denoted by $c_t^i$ on the amount of collateral posted by party $i$, which is not necessarily equal to the risk-free rate of the same currency, $r_t^i$. The fractional recovery rate $R_t^i \in [0, 1]$ is assumed for the under- as well as over-collateralized exposure. For the both parties, totally inaccessible default is assumed, and the hazard rate process of party $i$ is denoted by $h_t^i$. We assume there is no simultaneous default of the party 1 and 2, almost surely. Then, the pre-default value $V_t$ of the contract from the view point of party 1 is given by*

$$V_t = E^Q \left[ \int_{]t,T]} \exp\left( -\int_t^s (r_u - \mu(u, V_u)) du \right) dD_s \middle| \mathcal{F}_t \right], \quad t \leq T \quad (2.6)$$

*where*

$$\mu(t, v) = \left( y_t^1 \delta_t^1 - (1 - R_t^1)(1 - \delta_t^1)^+ h_t^1 + (1 - R_t^2)(\delta_t^1 - 1)^+ h_t^2 \right) \mathbf{1}_{\{v<0\}}$$
$$+ \left( y_t^2 \delta_t^2 - (1 - R_t^2)(1 - \delta_t^2)^+ h_t^2 + (1 - R_t^1)(\delta_t^2 - 1)^+ h_t^1 \right) \mathbf{1}_{\{v\geq 0\}} \quad (2.7)$$

*if the jump of $V$ at the time of default ($= \tau$) is zero almost surely, and then satisfies $S_t = V_t \mathbf{1}_{\{\tau>t\}}$ for all $t$. Here, $S_t$ is defined in Eq. (2.3).*

See Appendix A for proof. One important point regarding to this result is the fact that we can actually determine $y^i$ almost uniquely from the information of cross currency market. This point will be discussed in Sec. 6.4.

*Remark*: In this remark, we briefly discuss the assumption of $\Delta V_\tau = 0$. Notice that, since we assume totally inaccessible default time, there is no contribution from pre-fixed lump-sum coupon payments to the jump. In addition, it is natural (and also common in the existing literatures) to assume global market variables, such as interest rates and FX's, are adapted to the background filtration independent from the defaults. In this paper, we are concentrating on the standard fixed income derivatives without credit sensitive dividends, and hence the only thing we need to care about is the behavior of hazard rates, $h^1$ and $h^2$. Therefore, in this case, if there is no jump on $h^i$ on the default of the other party $j \neq i$, then the assumption $\Delta V_\tau = 0$ holds true. This corresponds to the situation where there is no default dependence between the two firms.

If there exists non-zero default dependence, which is important in risk-management point of view, then there appears a jump on the hazard rate of the surviving firm when a default occurs. This represents a direct feedback (or a contagious effect) from the defaulted firm to the surviving one. In this case, if we directly use $\mathbb{F}$-intensities $h^i$, the no-jump assumption does not hold.

However, even in this case, there is a way to handle the pricing problem correctly. Let us construct the filtration in the usual way as $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}^1_t \vee \mathcal{H}^2_t$, where $\mathcal{G}_t$ is the background filtration (say, generated by Brownian motions), and $\mathcal{H}^i_t$ is the filtration generated by $H^i$. Since the only information we need is up to $\tau = \tau^1 \wedge \tau^2$, we can limit our attention to the intensities conditional on no-default, which are now the processes adapted to the background filtration $\mathbb{G} = (\mathcal{G}_t)_{\{t \geq 0\}}$. Therefore, although the details of the derivation slightly change, one can show that the pricing formula given in Eq. (2.6) can still be applied in the same way once we use the $\mathbb{G}$-intensities instead, since now we can write all the processes involved in the formula adapted to the background filtration.

## 3 Symmetric Collateralization

Let us define

$$\tilde{y}^i_t = \delta^i_t y^i_t - (1 - R^i_t)(1 - \delta^i_t)^+ h^i_t + (1 - R^j_t)(\delta^i_t - 1)^+ h^j_t, \tag{3.1}$$

where $i, j \in \{1, 2\}$ and $j \neq i$. In the case of $\tilde{y}^1_t = \tilde{y}^2_t = \tilde{y}_t$, we have $\mu(t, V_t) = \tilde{y}_t$ that is independent from the contract value $V_t$. Therefore, from Proposition 1, we have

$$V_t = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \tilde{y}_u) du\right) dD_s \middle| \mathcal{F}_t\right] . \tag{3.2}$$

It is clear that simple redefinition of discounting rate allows us to evaluate a contract value in a standard way. Now, let us consider some important examples of symmetric and perfect collateralization where ($y^1 = y^2$) and ($\delta^1 = \delta^2 = 1$). One can easily confirm that all the following results are consistent with those given in Refs. [7, 8, 10, 9].

**Case 1**: Situation where both parties use the same collateral currency "$(i)$", which is the same as the payment currency. In this case, the pre-default value of the contract in terms of currency $(i)$ is given by

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s c_u^{(i)} du\right) dD_s \middle| \mathcal{F}_t\right] , \tag{3.3}$$

where $Q^{(i)}$ is the spot-martingale measure of currency $(i)$.

**Case 2**: Situation where both parties use the same collateral currency "$(k)$", which is the different from the payment currency "$(i)$". In this case, the pre-default value of the contract in terms of currency $(i)$ is given by

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + y_u^{(i,k)}) du\right) dD_s \middle| \mathcal{F}_t\right] , \tag{3.4}$$

where we have defined

$$\begin{aligned} y_u^{(i,k)} &= y_u^{(i)} - y_u^{(k)} & (3.5)\\ &= \left(r_u^{(i)} - c_u^{(i)}\right) - \left(r_u^{(k)} - c_u^{(k)}\right) . & (3.6) \end{aligned}$$

**Case 3**: Situation where the payment currency is $(i)$ and both parties optimally choose a currency from a common set of eligible collaterals denoted by $\mathcal{C}$ in each time they post collateral. In this case,

$$V_t^{(i)} = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + \max_{k\in\mathcal{C}} y_u^{(i,k)}) du\right) dD_s \middle| \mathcal{F}_t\right] \tag{3.7}$$

gives the pre-default value of the contract in terms of currency $(i)$. Note that collateral payer chooses currency $(k)$ that maximizes the effective discounting rate in order to reduce the mark-to-market loss. This is also the currency with the cheapest funding cost. See Sec. 6.4 and its *Remark* for details.

*Remark*: Notice that we have recovered linearity of each payment on the pre-default value for all these cases. In fact, in the case of symmetric collateralization, we can value the portfolio by adding the contribution from each trade/payment separately. This point can be considered as a good advantage of symmetric collateralization for practical use, since it makes agreement among financial firms easier as the transparent benchmark price in the market.

## 4 Marginal Impact of Asymmetry

We now consider more generic cases. When $\tilde{y}_t^1 \neq \tilde{y}_t^2$, we have non-linearity (called semi-linear in particular) in effective discounting rate $R(t, V_t) = r_t - \mu(t, V_t)$. Although it is possible to get solution by solving PDE in principle, it will soon become infeasible as the underlying dimension increases. Even if we adopt a very simple dynamic model, usual "reset advance pay arrear" conventions easily make the issue very complicated to handle.

For practical and feasible analysis, we use Gateaux derivative that was introduced in Duffie & Huang [3] to study the effects of default-spread asymmetry. We can follow the same procedure by appropriately redefining the variables. Since evaluation is straightforward in a symmetric case, the expansion of the pre-default value around the symmetric limit allows us simple analytic and/or numerical treatment. Firstly, let us define the spread process:

$$\tilde{\eta}_t^{i,j} = \tilde{y}_t^i - \tilde{y}_t^j . \tag{4.1}$$

Then, under an assumption that $\tilde{y}^i$ and $\tilde{y}^j$ do not depend on $V$ directly, the first-order effect on the pre-default value due to the non-zero spread appears as the following Gateaux derivative (See, Ref. [5] for details.):

$$\nabla V_t(0;\tilde{\eta}^{2,1}) = E^Q\left[\int_t^T e^{-\int_t^s (r_u - \tilde{y}_u^2)du} \max(-V_s(0),0)\tilde{\eta}_s^{2,1} ds \middle| \mathcal{F}_t\right] , \tag{4.2}$$

where $V_t(0)$ is the pre-default value of contract at time $t$ with the limit of $\tilde{\eta}^{2,1} \equiv 0$ and given by

$$V_t(0) = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - \tilde{y}_u^2)du\right) dD_s \middle| \mathcal{F}_t\right] . \tag{4.3}$$

Then the original pre-default value is approximated as

$$V_t \simeq V_t(0) + \nabla V_t(0;\tilde{\eta}^{2,1}) . \tag{4.4}$$

## 4.1 Asymmetric Collateralization

We now consider two special cases under perfect collateralization $\delta^1 = \delta^2 = 1$ using the previous result.

**Case 1**: The situation where the party 2 can only use the single collateral currency $(j)$ but party 1 chooses the optimal currency from the eligible set denoted by $\mathcal{C}$. The evaluation currency is $(i)$. In this case, the Gateaux derivative is given by

$$\nabla V_t(0;\max_{k\in C} y^{(j,k)}) = E^{Q^{(i)}}\left[\int_t^T \exp\left(-\int_t^s (c_u^{(i)} + y_u^{(i,j)})du\right) \max(-V_s(0),0) \max_{k\in C} y_s^{(j,k)} \middle| \mathcal{F}_t\right] , \tag{4.5}$$

where

$$V_t(0) = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + y_u^{(i,j)})du\right) dD_s \middle| \mathcal{F}_t\right] , \tag{4.6}$$

which is straightforward to calculate. This case is particularly interesting since the situation can naturally arise if the sophistication of collateral management of one of the parties is not enough to carry out optimal strategy, even when the relevant CSA is actually symmetric. We will carry out numerical study for this example in Sec. 7.

**Case 2**: The case of unilateral collateralization, where the party 2 is default-free and do not post collateral. The party 1 needs to post collateral in currency $(j)$ to fully cover the exposure, or $\delta^1 = 1$. The evaluation currency is $(i)$. We expand the pre-default value around the symmetric collateralization with currency $(j)$. In this case,

$$R(t,V_t) = r_t^{(i)} - y_t^{(j)}\mathbf{1}_{\{V_t<0\}} = c_t^{(i)} + y_t^{(i,j)} + y_t^{(j)}\mathbf{1}_{\{V_t\geq 0\}}.$$

$$\nabla V_t\big(0; y_t^{(j)}\mathbf{1}_{\{V_t\geq 0\}}\big) = -E^{Q^{(i)}}\left[\int_t^T \exp\left(-\int_t^s (c_u^{(i)} + y_u^{(i,j)})du\right)\max\big(V_s(0),0\big)y_s^{(j)}ds\,\Big|\,\mathcal{F}_t\right], \tag{4.7}$$

where

$$V_t(0) = E^{Q^{(i)}}\left[\int_{]t,T]}\exp\left(-\int_t^s (c_u^{(i)} + y_u^{(i,j)})du\right)dD_s\,\Big|\,\mathcal{F}_t\right] \tag{4.8}$$

is the value in symmetric limit. Detailed implications for the one-way CSA will be discussed in a later section after considering remaining credit risk.

In both cases, the correction term seems a weighted average of European option on the underlying contract. If we have analytic formula for $V_t(0)$, it is straightforward to carry out numerical calculation. The important factors determining the correction term are the dynamics of $y$ and $V$ itself, and their interdependence. This point will be studied in later sections.

# 5 CVA as a Deviation from Perfect Collateralization

As another important application of Gateaux derivative, we can consider CVA as a deviation from the perfect collateralization. Most of the existing literature is neglecting the cost of collateral for the calculation of CVA, which seems inappropriate considering the significant size and volatility of $y$, pointed out in our work [10] [2].

## 5.1 Derivation of CVA

Let us suppose $y_t^1 = y_t^2 = y_t$ for simplicity. In this case, we have

$$\begin{aligned}\mu(t,V_t) &= y_t - \Big((1-\delta_t^1)y_t + (1-R_t^1)(1-\delta_t^1)^+h_t^1 - (1-R_t^2)(\delta_t^1-1)^+h_t^2\Big)\mathbf{1}_{\{V_t<0\}} \\ &\quad -\Big((1-\delta_t^2)y_t + (1-R_t^2)(1-\delta_t^2)^+h_t^2 - (1-R_t^1)(\delta_t^2-1)^+h_t^1\Big)\mathbf{1}_{\{V_t\geq 0\}} \end{aligned}\tag{5.1}$$

and consider the Gateaux derivative around the point of $\delta^1 = \delta^2 = 1$. The result can be interpreted as a bilateral CVA that takes into account the cost of collateral and its coverage ratio explicitly. There also appears a new term "CCA" (collateral cost adjustment) that is purely the adjustment of collateral cost totally independent from the counterparty credit risk.

Following the method given in Ref. [5], one obtains

$$\begin{aligned}\nabla V_t &= E^Q\left[\int_{]t,T]} e^{-\int_t^s (r_u - y_u)du}(-V_s(0))\times\right. \\ &\quad \Big[\Big\{(1-\delta_s^1)y_s + (1-R_s^1)(1-\delta_s^1)^+h_s^1 - (1-R_s^2)(\delta_s^1-1)^+h_s^2\Big\}\mathbf{1}_{\{V_s(0)<0\}} \\ &\quad \left.+\Big\{(1-\delta_s^2)y_s + (1-R_s^2)(1-\delta_s^2)^+h_s^2 - (1-R_s^1)(\delta_s^2-1)^+h_s^1\Big\}\mathbf{1}_{\{V_s(0)\geq 0\}}\Big]\,\Big|\,\mathcal{F}_t\right], \end{aligned}\tag{5.2}$$

[2]For general treatment of CVA and related references, see Ref. [1], for example.

where

$$V_t(0) = E^Q\left[\int_{]t,T]} \exp\left(-\int_t^s (r_u - y_u)du\right) dD_s \middle| \mathcal{F}_t\right] , \tag{5.3}$$

which represents the contract value under the perfect collateralization. Using the above result, the contract value can be decomposed into three parts, one is the value under the perfect collateralization, CCA (collateral cost adjustment) and CVA [3].

$$V_t \simeq V_t(0) + \text{CCA} + \text{CVA} . \tag{5.4}$$

This decomposition would be useful for practitioners who know that most of their exposure is collateralized, but still care about the remaining small counter party exposure and adjustment of collateral cost due to the deviation from the perfect collateralization [4]. It is natural to expand around the perfectly collateralized limit, since it would be the only choice that can achieve the required transparency as the benchmark price in the market. By expanding Eq.(5.2), we have

$$\text{CCA} \;=\; E^Q\left[\int_t^T e^{-\int_t^s (r_u-y_u)du} y_s\Big[(1-\delta_s^1)\big[-V_s(0)\big]^+ - (1-\delta_s^2)\big[V_s(0)\big]^+\Big]ds \middle| \mathcal{F}_t\right] \tag{5.5}$$

which is a pure adjustment of collateral cost due to the deviation from the perfect collateralization, and independent from the credit risk.

For credit sensitive part, we have

$$\begin{aligned}
\text{CVA} \;=&\\
E^Q&\left[\int_{]t,T]} e^{-\int_t^s (r_u-y_u)du}(1-R_s^1)h_s^1\Big\{(1-\delta_s^1)^+\big[-V_s(0)\big]^+ + (\delta_s^2-1)^+\big[V_s(0)\big]^+\Big\}ds \middle| \mathcal{F}_t\right]\\
-E^Q&\left[\int_{]t,T]} e^{-\int_t^s (r_u-y_u)du}(1-R_s^2)h_s^2\Big\{(1-\delta_s^2)^+\big[V_s(0)\big]^+ + (\delta_s^1-1)^+\big[-V_s(0)\big]^+\Big\}ds \middle| \mathcal{F}_t\right] .
\end{aligned} \tag{5.6}$$

The effects of stochastic coverage ratio as well as non-zero jump at the time of default are our ongoing research topics.

## 5.2 Implications of Collateralization to Price Adjustment

Although we leave detailed numerical study of CVA under collateralization for a separate paper, let us make several qualitative observations here. Firstly, although the terms in CVA are pretty similar to the usual result of bilateral CVA, the discounting rate is now different from the risk-free rate and reflects the funding cost of collateral. If there is no dependency between $y$ and other variables, such as hazard rate, the effects of collateralization would mainly appear through the modification of discounting factor. As we have

[3] Our convention of CVA is different from other literatures by sign where it is defined as the "charge" to the clients. Thus, $\text{CVA}_{\text{ours}} = -\text{CVA}$.

[4] One can perform the same procedures even if there exist asymmetry in collateralization. Since we expand around symmetric limit, there also appears correction terms for asymmetry.

studied in Ref. [10], the change of effective discounting factor due to the choice of collateral currency or optimal collateral strategy can be as big as several tens of percentage points. This itself can modify the resultant CVA meaningfully. In the case of correlated $y$ and other variables, particularly the hazard rates, there may appear new type of "wrong way" risk. As we will see later, $y$ is closely related to the CCS basis spread that reflects the relative funding cost difference between the two currencies involved. Hence, $y$ is expected to be highly sensitive to the market liquidity, and hence is also strongly affected by the overall market credit conditions. Therefore, although the efficient collateral management significantly reduce the credit risk, one needs to carefully estimate the remaining credit exposures when there exists a meaningful deviation from the perfect collateralization.

Secondly, we can also expect important effects from the stochastic coverage ratios. If the main reason for the imperfectness of collateralization comes from price disputes over exotic products, $\delta^i$ may be well regressed by market skewness, volatility level, Libor-OIS and CCS basis spreads, etc. This may create non-trivial dependence among the collateral coverage ratio, the credit exposure, and also on the funding cost of collateral. By monitoring the price disagreements, financial firms should be able to construct a realistic model of $\delta^i$ for each counter party. It will be also useful for stress testing allowing higher dependence among them.

Thirdly, as we have seen, there appears a new term called "CCA" which adjusts the cost of collateral from the perfect collateralization case. Dependent on the details of contracts and correlation among the underlying variables, CCA can be as important as CVA. As can be seen from Eq. (5.5), it will be particularly the case when there is significant correlation between the collateral cost $y$ and the underlying contract value. A typical examples of the products highly correlated with $y$ are cross currency basis swap and probably sovereign risk sensitive products.

As the last remark, the valuation of CVA is critically depend on the recovery or closeout scheme in general, and the result may sometimes be counterintuitive and/or inappropriate, as clearly demonstrated by the recent work of Brigo & Morini (2010) [2]. However, in the case of a collateralized contract, the dependency on the closeout conventions is expected to be quite small. This is because, the creditworthiness of both parties which enter the substitution trade is largely flattened by collateralization.

### 5.3 Several special cases for CVA

Let us consider several important examples:
**Case 1**: Consider the situation where the both parties use collateral currency $(i)$, which is the same as the payment currency. We also assume a common constant coverage ratio $\delta^1 = \delta^2 = \delta\ (< 1)$, and also constant recovery rates. In this case, CCA and CVA are given by

$$\mathrm{CCA} = -(1-\delta)E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s c_u^{(i)}du} y_s^{(i)} V_s(0)ds \middle| \mathcal{F}_t\right] \tag{5.7}$$

$$\begin{aligned}\mathrm{CVA} = {} & (1-R^1)(1-\delta)E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s c_i^{(i)}du} h_s^1 \max\big(-V_s(0),0\big)ds \middle| \mathcal{F}_t\right] \\ & -(1-R^2)(1-\delta)E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s c_u^{(i)}du} h_s^2 \max\big(V_s(0),0\big)ds \middle| \mathcal{F}_t\right] ,\end{aligned} \tag{5.8}$$

where

$$V_t(0) = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s c_u^{(i)} du\right) dD_s \middle| \mathcal{F}_t\right] \tag{5.9}$$

is a value under perfect collateralization by domestic currency.

**Case 2**: Consider the situation where the both parties optimally choose collateral currency $(k)$ from the eligible collateral set $\mathcal{C}$. The payments are done by currency $(i)$. We assume the common constant coverage ration $\delta$ $(< 1)$ and constant recovery rates. In this case, CCA and CVA are given by

$$\begin{aligned} \mathrm{CCA} &= -(1-\delta)E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s (c_u^{(i)} + \max_{k\in\mathcal{C}} y_u^{(i,k)})du} y_s^{(k)} V_s(0) ds \middle| \mathcal{F}_t\right] \qquad (5.10) \\ \mathrm{CVA} &= +(1-R^1)(1-\delta)E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s (c_u^{(i)} + \max_{k\in\mathcal{C}} y_u^{(i,k)})du} h_s^1 \max\big(-V_s(0),0\big) ds \middle| \mathcal{F}_t\right] \\ &\quad -(1-R^2)(1-\delta)E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s (c_u^{(i)} + \max_{k\in\mathcal{C}} y_u^{(i,k)})du} h_s^2 \max\big(V_s(0),0\big) ds \middle| \mathcal{F}_t\right], \qquad (5.11) \end{aligned}$$

where

$$V_t(0) = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s (c_u^{(i)} + \max_{k\in\mathcal{C}} y_u^{(i,k)})\right) dD_s \middle| \mathcal{F}_t\right]. \tag{5.12}$$

An interesting point is that the optimal choice of collateral currency may significantly change the size of CVA relative to the single currency case due to the increase of effective discounting rates as discovered in Ref. [10].

**Case 3**: Let us consider another important situation, which is the unilateral collateralization with bilateral default risk. Suppose the situation where only the party 2 is required to post collateral due to its high credit risk relative to the party 1. We have $\delta^1 = 0$, $\delta^2 \simeq 1$, and write $y_t^2 = y_t$. In this case we have

$$\begin{aligned} \mu(t, V_t) &= y_t - \left[y_t \mathbf{1}_{\{V_t<0\}} + (1-\delta_t^2) y_t \mathbf{1}_{\{V_t\geq 0\}}\right] \\ &\quad -(1-R_t^1)h_t^1\left(\mathbf{1}_{\{V_t<0\}} - (\delta_t^2-1)^+\mathbf{1}_{\{V_t\geq 0\}}\right) - (1-R_t^2)(1-\delta_t^2)^+ h_t^2 \mathbf{1}_{\{V_t\geq 0\}}. \qquad (5.13) \end{aligned}$$

Taking Gateaux derivative around the point of $\mu(t, V_t) = y_t$, we have

$$\begin{aligned} \nabla V_t &= E^Q\left[\int_t^T e^{-\int_t^s (r_u - y_u)du}\Big(-V_s(0)\Big)\times \right. \\ &\quad \Big[y_s \mathbf{1}_{\{V_s<0\}} + (1-\delta_s^2) y_s \mathbf{1}_{\{V_s\geq 0\}} + (1-R_s^1)h_s^1\big(\mathbf{1}_{\{V_s<0\}} - (\delta_s^2-1)^+\mathbf{1}_{\{V_s\geq 0\}}\big) \\ &\quad \left. + (1-R_s^2)(1-\delta_s^2)^+ h_s^2 \mathbf{1}_{\{V_s\geq 0\}}\Big] \middle| \mathcal{F}_t\right]. \qquad (5.14) \end{aligned}$$

More specifically, if we assume the same collateral and payment currency $(i)$, we have

$$V_t \simeq V_t(0) + \mathrm{CCA} + \mathrm{CVA}, \tag{5.15}$$

where

$$V_t(0) = E^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s c_u^{(i)}du\right) dD_s \middle| \mathcal{F}_t\right] \tag{5.16}$$

and

$$\begin{aligned}\text{CCA} &= E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s c_u^{(i)}du} y_s^{(i)} \left\{[-V_s(0)]^+ - (1-\delta_s^2)[V_s(0)]^+\right\} ds \middle| \mathcal{F}_t\right] \\ &\qquad (5.17) \\ \text{CVA} &= E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s c_u^{(i)}du}(1-R_s^1)h_s^1 \left\{[-V_s(0)]^+ + (\delta_s^2 - 1)^+[V_s(0)]^+\right\} ds \middle| \mathcal{F}_t\right] \\ &\quad - E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s c_u^{(i)}du}(1-R_s^2)(1-\delta_s^2)^+ h_s^2 [V_s(0)]^+ ds \middle| \mathcal{F}_t\right] \qquad (5.18)\end{aligned}$$

If party 1 receives "strong" currency (that is the currency with high value of $y^{(i)}$), such as USD (See, Ref. [10]), and also imposes stringent collateral management $\delta^2 \simeq 1$ on the counter party, it can enjoy significant funding benefit from CCA. The CVA terms are usual bilateral credit risk adjustment except that the discounting is now given by the collateral rate.

Note that, this example is particularly common when SSA (sovereign, supranational and agency) is involved (as party 1). For example, when the party 1 is a central bank, it does not post collateral but receives it. From the view point of the counterpart financial firm (party 2), this is a real headache. As we have explained in the introduction, since party 2 has to enter bilateral collateralization when it tries to hedge the position in the market, there clearly exists a significant risk of cash-flow mismatch. In addition, although the contribution from the CVA will be negligible, there exists a big mark-to-market issue from the CCA term. Even if it is not a critical matter at the current low-interest rate market, once the market interest rate starts to go up while the overnight rate $c$ is kept low by the central bank to support economy, the resultant mark-to-market loss for the party 2 can be quite significant due to the rising cost of collateral "$y$" (Remember that $y^{(i)} = r^{(i)} - c^{(i)}$).

**Case 4**: Finally, let us consider the situation where there exist collateral thresholds. A threshold is a level of exposure below which collateral will not be called, and hence it represents an amount of uncollateralized exposure. If the exposure is above the threshold, only the incremental exposure will be collateralized. Usually, the collateral thresholds are set according to the credit standing of each counter party. They are often asymmetric, with lower-rated counter party having a lower threshold than the higher-rated counter party. It may be adjusted according to the "triggers" linked to the credit rating during the contract. We assume that the threshold of counter party $i$ is set by $\Gamma_t^i > 0$, and that the exceeding exposure is perfectly collateralized continuously.

In this case, Eq. (2.3) is modified in the following way:

$$S_t = \beta_t E^Q \left[ \int_{]t,T]} \beta_u^{-1} \mathbf{1}_{\{\tau > u\}} \left\{ dD_u + q(u, S_u) S_u du \right\} + \int_{]t,T]} \beta_u^{-1} \mathbf{1}_{\{\tau \geq u\}} \left\{ Z^1(u, S_{u-}) dH_u^1 + Z^2(u, S_{u-}) dH_u^2 \right\} \middle| \mathcal{F}_t \right] , \quad (5.19)$$

where

$$q(t, S_t) = y_t^1 \left( 1 + \frac{\Gamma_t^1}{S_t} \right) \mathbf{1}_{\{S_t < -\Gamma_t^1\}} + y_t^2 \left( 1 - \frac{\Gamma_t^2}{S_t} \right) \mathbf{1}_{\{S_t \geq \Gamma_t^2\}} , \quad (5.20)$$

and

$$\begin{aligned} Z^1(t, S_t) &= S_t \left[ \left( 1 + (1 - R_t^1) \frac{\Gamma_t^1}{S_t} \right) \mathbf{1}_{\{S_t < -\Gamma_t^1\}} + R_t^1 \mathbf{1}_{\{-\Gamma_t^1 \leq S_t < 0\}} + \mathbf{1}_{\{S_t \geq 0\}} \right] \\ Z^2(t, S_t) &= S_t \left[ \left( 1 - (1 - R_t^2) \frac{\Gamma_t^2}{S_t} \right) \mathbf{1}_{\{S_t \geq \Gamma_t^2\}} + R_t^2 \mathbf{1}_{\{0 \leq S_t < \Gamma_t^2\}} + \mathbf{1}_{\{S_t < 0\}} \right] . \end{aligned}$$

Here, we have assumed the same recovery rate for the uncollateralized exposure regardless of whether the contract value is above or below the threshold.

Following the same procedures given in Appendix A, one can show that the pre-default value of the contract $V_t$ is given by

$$V_t = E^Q \left[ \int_{]t,T]} \exp \left( - \int_t^s (r_u - \mu(u, V_u)) du \right) dD_s \middle| \mathcal{F}_t \right] , \quad t \leq T \quad (5.21)$$

where

$$\begin{aligned} \mu(t, V_t) &= y_t^1 \mathbf{1}_{\{V_t < 0\}} + y_t^2 \mathbf{1}_{\{V_t \geq 0\}} \\ &\quad - \left( y_t^1 + h_t^1 (1 - R_t^1) \right) \left[ \mathbf{1}_{\{-\Gamma_t^1 \leq V_t < 0\}} - \frac{\Gamma_t^1}{V_t} \mathbf{1}_{\{V_t < -\Gamma_t^1\}} \right] \\ &\quad - \left( y_t^2 + h_t^2 (1 - R_t^2) \right) \left[ \mathbf{1}_{\{0 \leq V_t < \Gamma_t^2\}} + \frac{\Gamma_t^2}{V_t} \mathbf{1}_{\{V_t \geq \Gamma_t^2\}} \right] . \end{aligned} \quad (5.22)$$

Now, consider the case where the both parties use the same collateral currency $(i)$, which is equal to the evaluation currency of the contract. Then, we have

$$\begin{aligned} \mu(t, V_t) &= y_t^{(i)} - \left\{ y_t^{(i)} \mathbf{1}_{\{-\Gamma_t^1 \leq V_t < \Gamma_t^2\}} \right. \\ &\quad + y_t^{(i)} \left[ \frac{\Gamma_t^1}{V_t} \mathbf{1}_{\{V_t < -\Gamma_t^1\}} - \frac{\Gamma_t^2}{V_t} \mathbf{1}_{\{V_t \geq \Gamma_t^2\}} \right] \\ &\quad - h_t^1 (1 - R_t^1) \left[ \mathbf{1}_{\{-\Gamma_t^1 \leq V_t < 0\}} - \frac{\Gamma_t^1}{V_t} \mathbf{1}_{\{V_t < -\Gamma_t^1\}} \right] \\ &\quad \left. - h_t^2 (1 - R_t^2) \left[ \mathbf{1}_{\{0 \leq V_t < \Gamma_t^2\}} + \frac{\Gamma_t^2}{V_t} \mathbf{1}_{\{V_t \geq \Gamma_t^2\}} \right] \right\} . \end{aligned} \quad (5.23)$$

Hence, if we apply Gateaux derivative around the symmetric perfect collateralization with currency $(i)$ that is $\mu(t, V_t) = y_t^{(i)}$, we obtain

$$V_t \simeq V_t(0) + \text{CCA} + \text{CVA}, \tag{5.24}$$

where

$$V_t(0) = \mathbf{E}^{Q^{(i)}}\left[\int_{]t,T]} \exp\left(-\int_t^s c_u^{(i)} du\right) dD_s \middle| \mathcal{F}_t\right], \tag{5.25}$$

and

$$\begin{aligned}
\text{CCA} &= -E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s c_u^{(i)} du} y_s^{(i)} V_s(0) \mathbf{1}_{\{-\Gamma_s^1 \le V_s(0) < \Gamma_s^2\}} ds \middle| \mathcal{F}_t\right] \\
&\quad + E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s c_u^{(i)} du} y_s^{(i)} \left[\Gamma_s^1 \mathbf{1}_{\{V_s(0) < -\Gamma_s^1\}} - \Gamma_s^2 \mathbf{1}_{\{V_s(0) \ge \Gamma_s^2\}}\right] ds \middle| \mathcal{F}_t\right] \qquad (5.26) \\
\text{CVA} &= -E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s c_u^{(i)} du}\left[h_s^1(1 - R_s^1)\left(V_s(0)\mathbf{1}_{\{-\Gamma_s^1 \le V_s(0) < 0\}} - \Gamma_s^1 \mathbf{1}_{\{V_s(0) < -\Gamma_s^1\}}\right)\right] ds \middle| \mathcal{F}_t\right] \\
&\quad - E^{Q^{(i)}}\left[\int_t^T e^{-\int_t^s c_u^{(i)} du}\left[h_s^2(1 - R_s^2)\left(V_s(0)\mathbf{1}_{\{0 < V_s(0) \le \Gamma_s^2\}} + \Gamma_s^2 \mathbf{1}_{\{V_s(0) > \Gamma_s^2\}}\right)\right] ds \middle| \mathcal{F}_t\right]. \qquad (5.27)
\end{aligned}$$

It is easy to see that the terms in CCA are reflecting the fact that no collateral is being posted in the range $\{-\Gamma_t^1 \le V_t \le \Gamma_t^2\}$, and that the posted amount of collateral is smaller than $|V|$ by the size of threshold. The terms in CVA represent bilateral uncollateralized credit exposure, which is capped by each threshold.

## 6 Fundamental Instruments

In order to study the quantitative effects of collateralization and its implications on CVA, we need to understand the pricing of fundamental instruments under symmetric collateralization. It is also necessary for the calibration of the model in the first place. One obtains detailed discussion in Refs [7, 8, 10], but we extend the results for stochastic $y$ spread and summarize in this section. We also introduce a slightly simpler cross currency swap, which is actually tradable in the market, in order to show the direct link of CCS with the cost of collateral in much simpler fashion. All the results easily follow from Sec. 3.

Throughout this section, we assume that the market quotes of standard products are the values under symmetric and perfect collateralization, which should be reasonable considering dominant role of major broker-dealers for these products and their stringent collateral management. If it is not the case, value of any contract becomes dependent on the portfolio to a specific counter party, which makes it impossible for financial firms to agree on the market prices. In fact, to achieve enough transparency in the market quotes, the broker-dealers should specify the details of CSA to avoid contamination from contracts with non-standard collateral agreements.

## 6.1 Collateralized Zero Coupon Bond

A collateralized zero coupon bond is the most fundamental asset for the valuation of all the contracts with collateral agreements. We denote a zero coupon bond collateralized by the domestic currency $(i)$ as

$$D^{(i)}(t,T) = E^{Q^{(i)}}\left[ e^{-\int_t^T c_s^{(i)} ds} \middle| \mathcal{F}_t \right] \tag{6.1}$$

If payment and collateralized currencies are different, $(i)$ and $(j)$ respectively, a foreign collateralized zero coupon bond $D^{(i,j)}$ is given by

$$D^{(i,j)}(t,T) = E^{Q^{(i)}}\left[ e^{-\int_t^T c_s^{(i)} ds} \left( e^{-\int_t^T y_s^{(i,j)} ds} \right) \middle| \mathcal{F}_t \right] . \tag{6.2}$$

In particular, if $c^{(i)}$ and $y^{(i,j)}$ are independent, we have

$$D^{(i,j)}(t,T) = D^{(i)}(t,T) e^{-\int_t^T y^{(i,j)}(t,s) ds} , \tag{6.3}$$

where

$$y^{(i,j)}(t,s) = -\frac{1}{s} \ln E^{Q^{(i)}}\left[ e^{-\int_t^s y_u^{(i,j)} du} \middle| \mathcal{F}_t \right] \tag{6.4}$$

denotes the forward $y^{(i,j)}$ spread.

## 6.2 Collateralized FX Forward

Because of the existence of collateral, FX forward transaction now becomes non-trivial. The precise understanding of the collateralized FX forward is crucial to deal with generic collateralized products.

The definition of currency-$(k)$ collateralized FX forward contract for the currency pair $(i,j)$ is as follows:

• *At the time of trade inception $t$, both parties agree to exchange $K$ unit of currency $(i)$ with the one unit of currency $(j)$ at the maturity $T$. Throughout the contract period, the continuous collateralization by currency $(k)$ is performed, i.e. the party who has negative mark-to-market value need to post the equivalent amount of cash in currency $(k)$ to the counter party as collateral, and this adjustment is made continuously. FX forward rate $f_x^{(i,j)}(t,T;k)$ is defined as the value of $K$ that makes the value of contract at the inception time zero.*

By using the results of Sec. 3, $K$ needs to satisfy the following relation:

$$K E^{Q^{(i)}}\left[ e^{-\int_t^T \left( c_s^{(i)} + y_s^{(i,k)} \right) ds} \middle| \mathcal{F}_t \right] - f_x^{(i,j)}(t) E^{Q^{(j)}}\left[ e^{-\int_t^T \left( c_s^{(j)} + y_s^{(j,k)} \right) ds} \middle| \mathcal{F}_t \right] = 0 \tag{6.5}$$

and hence the FX forward is given by

$$f_x^{(i,j)}(t,T;k) = f_x^{(i,j)}(t)\frac{E^{Q^{(j)}}\left[e^{-\int_t^T\left(c_s^{(j)}+y_s^{(j,k)}\right)ds}\Big|\mathcal{F}_t\right]}{E^{Q^{(i)}}\left[e^{-\int_t^T\left(c_s^{(i)}+y_s^{(i,k)}\right)ds}\Big|\mathcal{F}_t\right]} \tag{6.6}$$

$$= f_x^{(i,j)}(t)\frac{D^{(j,k)}(t,T)}{D^{(i,k)}(t,T)}, \tag{6.7}$$

which becomes a martingale under the $(k)$-collateralized forward measure. In particular, we have

$$E^{Q^{(i)}}\left[e^{-\int_t^T\left(c_s^{(i)}+y_s^{(i,k)}\right)ds}f_x^{(i,j)}(T)\Big|\mathcal{F}_t\right] = D^{(i,k)}(t,T)E^{T^{(i,k)}}\left[f_x^{(i,j)}(T,T;k)\Big|\mathcal{F}_t\right]$$

$$= D^{(i,k)}(t,T)f_x^{(i,j)}(t,T;k)\ . \tag{6.8}$$

Here, we have defined the $(k)$-collateralized $(i)$ forward measure $T^{(i,k)}$, where $D^{(i,k)}(\cdot,T)$ is used as the numeraire. $E^{T^{(i,k)}}[\cdot]$ denotes expectation under this measure.

## 6.3 Overnight Index Swap

The overnight index swap (OIS) is a fixed-vs-floating swap which is the same as the usual IRS except that the floating leg pays periodically, say quarterly, daily compounded ON rate instead of Libors. Let us consider $T_0$-start $T_N$-maturing OIS of currency $(j)$ with fixed rate $S_N$, where $T_0 \geq t$. If the party 1 takes a receiver position, we have

$$dD_s = \sum_{n=1}^{N}\delta_{T_n}(s)\left[\Delta_n S_N + 1 - \exp\left(\int_{T_{n-1}}^{T_n}c_u^{(j)}du\right)\right] \tag{6.9}$$

where $\Delta$ is day-count fraction of the fixed leg, and $\delta_T(\cdot)$ denotes Dirac delta function at $T$.

Using the results of Sec. 3, in the case of currency $(k)$ collateralization, we have

$$V_t^{(j)} = E^{Q^{(j)}}\left[\int_{]T_0,T_N]}\exp\left(-\int_t^s(c_u^{(j)}+y_u^{(j,k)})du\right)dD_s\Big|\mathcal{F}_t\right] \tag{6.10}$$

$$= \sum_{n=1}^{N}E^{Q^{(j)}}\left[e^{-\int_t^{T_n}\left(c_u^{(j)}+y_u^{(j,k)}\right)du}\left(\Delta_n S_N + 1 - e^{\int_{T_{n-1}}^{T_n}c_u^{(j)}du}\right)\Big|\mathcal{F}_t\right]\ . \tag{6.11}$$

In particular, if OIS is collateralized by its domestic currency $(j)$, we have

$$V_t^{(j)} = \sum_{n=1}^{N}\Delta_n D^{(j)}(t,T_n)S_N - \left(D^{(j)}(t,T_0) - D^{(j)}(t,T_N)\right), \tag{6.12}$$

and hence the par rate is given by

$$S_N = \frac{D^{(j)}(t,T_0) - D^{(j)}(t,T_N)}{\sum_{n=1}^{N}\Delta_n D^{(j)}(t,T_n)}\ . \tag{6.13}$$

## 6.4 Cross Currency Swap

Cross currency swap (CCS) is one of the most fundamental products in FX market. Especially, for maturities longer than a few years, CCS is much more liquid than FX forward contract, which gives CCS a special role for model calibrations. The current market is dominated by USD crosses where 3m USD Libor flat is exchanged by 3m Libor of a different currency with additional (negative in many cases) basis spread. The most popular type of CCS is called MtMCCS (Mark-to-Market CCS) in which the notional of USD leg is rest at the start of every calculation period of Libor, while the notional of the other leg is kept constant throughout the contract period. For model calibration, MtMCCS should be used as we have done in Ref. [10] considering its liquidity. However, in this paper, we study a different type of CCS, which is actually tradable in the market, to make the link between $y$ and CCS much clearer.

We study the Mark-to-Market cross currency overnight index swap (MtMCCOIS), which is exactly the same as the usual MtMCCS except that it pays a compounded ON rate, instead of the Libor, of each currency periodically. Let us consider $(i,j)$-MtMCCOIS where currency $(i)$ intended to be USD and needs notional refreshments, and currency $(j)$ is the one in which the basis spread is to be paid. Let us suppose the party 1 is the spread receiver and consider $T_0$-start $T_N$-maturing $(i,j)$-MtMCCOIS. For the $(j)$-leg, we have

$$dD_s^{(j)} = -\delta_{T_0}(s) + \delta_{T_N}(s) + \sum_{n=1}^{N} \delta_{T_n}(s) \left[ \left( e^{\int_{T_{n-1}}^{T_n} c_u^{(j)} du} - 1 \right) + \delta_n B_N \right] , \qquad (6.14)$$

where $B_N$ is the basis spread of the contract. For $(i)$-leg, in terms of currency $(i)$, we have

$$dD_s^{(i)} = \sum_{n=1}^{N} \left[ \delta_{T_{n-1}}(s) f_x^{(i,j)}(T_{n-1}) - \delta_{T_n}(s) f_x^{(i,j)}(T_{n-1}) e^{\int_{T_{n-1}}^{T_n} c_u^{(i)} du} \right] . \qquad (6.15)$$

In total, in terms of currency $(j)$, we have

$$\begin{aligned} dD_s &= dD_s^{(j)} + f_x^{(j,i)}(s) dD_s^{(i)} & (6.16) \\ &= dD_s^{(j)} + \sum_{n=1}^{N} \left[ \delta_{T_{n-1}}(s) - \delta_{T_n}(s) \frac{f_x^{(j,i)}(T_n)}{f_x^{(j,i)}(T_{n-1})} e^{\int_{T_{n-1}}^{T_n} c_u^{(i)} du} \right] & (6.17) \\ &= \sum_{n=1}^{N} \delta_{T_n}(s) \left[ e^{\int_{T_{n-1}}^{T_n} c_u^{(j)} du} + \delta_n B_N - \frac{f_x^{(j,i)}(T_n)}{f_x^{(j,i)}(T_{n-1})} e^{\int_{T_{n-1}}^{T_n} c_u^{(i)} du} \right] . & (6.18) \end{aligned}$$

If the collateralization is done by currency $(k)$, then the value for the party 1 is given by

$$V_t = \sum_{n=1}^{N} E^{Q^{(j)}} \left[ e^{-\int_t^{T_n} (c_u^{(j)} + y_u^{(j,k)}) du} \left\{ e^{\int_{T_{n-1}}^{T_n} c_u^{(j)} du} + \delta_n B_N - \frac{f_x^{(j,i)}(T_n)}{f_x^{(j,i)}(T_{n-1})} e^{\int_{T_{n-1}}^{T_n} c_u^{(i)} du} \right\} \middle| \mathcal{F}_t \right] , \qquad (6.19)$$

where $T_0 \geq t$. In particular, let us consider the case where the swap is collateralized by currency ($i$) (or USD), which looks popular in the market.

$$
\begin{aligned}
V_t &= \sum_{n=1}^{N} \delta_n B_N D^{(j)}(t,T_n) e^{-\int_t^{T_n} y^{(j,i)}(t,u)du} \\
&\quad - \sum_{n=1}^{N} D^{(j)}(t,T_{n-1}) e^{-\int_t^{T_{n-1}} y^{(j,i)}(t,u)du} \left(1 - e^{-\int_{T_{n-1}}^{T_n} y^{(j,i)}(t,u)du}\right) \\
&= \sum_{n=1}^{N} \left[ \delta_n B_N D^{(j,i)}(t,T_n) - D^{(j,i)}(t,T_{n-1}) \left(1 - e^{-\int_{T_{n-1}}^{T_n} y^{(j,i)}(t,u)du}\right)\right] . \quad (6.20)
\end{aligned}
$$

Here, we have assumed the independence of $c^{(j)}$ and $y^{(j,i)}$. In fact, the assumption seems reasonable according to the recent historical data studied in Ref. [10]. In this case, we obtain the par MtMCCOIS basis spread as

$$
B_N = \frac{\sum_{n=1}^{N} D^{(j,i)}(t,T_{n-1}) \left(1 - e^{-\int_{T_{n-1}}^{T_n} y^{(j,i)}(t,u)du}\right)}{\sum_{n=1}^{N} \delta_n D^{(j,i)}(t,T_n)} . \quad (6.21)
$$

Thus, it is easy to see that

$$
B_N \sim \frac{1}{T_N - T_0} \int_{T_0}^{T_N} y^{(j,i)}(t,u)du, \quad (6.22)
$$

which gives us the relation between the relative difference of collateral cost $y^{(j,i)}$ and the observed cross currency basis. Therefore, the cost of collateral $y$ is directly linked to the dynamics of CCS markets.

*Remark: Origin of $y^{(i,j)}$ in Pricing Formula*
Here, let us comment about the origin of $y$ spread in our pricing formula in Proposition 1. Consider the following hypothetical but plausible situation to get a clear image:

*(1): An interest rate swap market where the participants are discounting future cash flows by domestic OIS rate, regardless of the collateral currency, and assume there is no price dispute among them. (2): Party 1 enters two opposite trades with party 2 and 3, and they are agree to have CSA which forces party 2 and 3 to always post a domestic currency $U$ as collateral, but party 1 is allowed to use a foreign currency $E$ as well as $U$. (3): There is very liquid CCOIS market which allows firms to enter arbitrary length of swap. The spread $y$ is negative for CCOIS between $U$ and $E$, where $U$ is a base currency (such as USD in the above explanation).*

In this example, the party 1 can definitely make money. Suppose, at a certain point, the party 1 receives $N$ unit amount of $U$ from the party 2 as collateral. Party 1 enters a CCOIS as spread payer, exchanging $N$ unit amount of $U$ and the corresponding amount of $E$, by which it can finance the foreign currency $E$ by the rate of ($E$'s OIS $+y$). Party 1 also receives $U$'s OIS rate from the CCOIS counter party, which is going to be paid as the collateral margin to the party 2. Party 1 also posts $E$ to the party 3 since it has opposite position, it receives $E$'s OIS rate as the collateral margin from the party 3. As a result, the party 1 earns $-y$ ($> 0$) on the notional amount of collateral. It can rollover the CCOIS, or unwind it if $y$'s sign flips.

Of course, in the real world, CCS can only be traded with certain terms which makes the issue not so simple. However, considering significant size of CCS spread (a several tens of bps) it still seems possible to arrange appropriate CCS contracts to achieve cheaper funding. For a very short term, it may be easier to use FX forward contracts for the same purpose. In order to prohibit this type of arbitrage, party 1 should pay extra premium to make advantageous CSA contracts. This is exactly the reason why our pricing formula contains the spread $y$.

### 6.5 Calibration to swap markets

For the details of calibration procedures, the numerical results and recent historical behavior of underlyings are available in Refs. [7, 10]. The procedures can be briefly summarized as follows: (1) Calibrate the forward collateral rate $c^{(i)}(0,t)$ for each currency using OIS market. (2) Calibrate the forward Libor curves by using the result of (1), IRS and tenor swap markets. (3) Calibrate the forward $y^{(i,j)}(0,t)$ spread for each relevant currency pair by using the results of (1),(2) and CCS markets.

Although we can directly obtains the set of $y^{(i,j)}$ from CCS, we cannot uniquely determine each $y^{(i)}$, which is necessary for the evaluation of Gateaux derivative when we deal with unilateral collateralization and CCA (collateral cost adjustment). For these cases, we need to make an assumption on the risk-free rate for one *and only one* currency. For example, if we assume that ON rate and the risk-free rate of currency $(j)$ are the same, and hence $y^{(j)} = 0$, then the forward curve of $y^{\mathrm{USD}}$ is fixed by $y^{\mathrm{USD}}(0,t) = -y^{(j,\mathrm{USD})}(0,t)$. Then using the result of $y^{\mathrm{USD}}$, we obtains $\{y^{(k)}\}$ for all the other currencies by making use of $\{y^{(k,\mathrm{USD})}\}$ obtained from CCS markets. More ideally, each financial firm may carry out some analysis on the risk-free profit rate of cash pool or more advanced econometric analysis on the risk-free rate, such as those given in Feldhütter & Lando (2008) [6].

## 7 Numerical Studies for Asymmetric Collateralization

In this section, we study the effects of asymmetric collateralization on the two fundamental products, MtMCCOIS and OIS, using Gateaux derivative. For both cases, we use the following dynamics in Monte Carlo simulation:

$$dc_t^{(j)} = \left(\theta^{(j)}(t) - \kappa^{(j)} c_t^{(j)}\right) dt + \sigma_c^{(j)} dW_t^1 \tag{7.1}$$

$$dc_t^{(i)} = \left(\theta^{(i)}(t) - \rho_{2,4}\sigma_c^{(i)}\sigma_x^{(j,i)} - \kappa^{(i)} c_t^{(i)}\right) dt + \sigma_c^{(i)} dW_t^2 \tag{7.2}$$

$$dy_t^{(j,i)} = \left(\theta^{(j,i)}(t) - \kappa^{(j,i)} y_t^{(j,i)}\right) dt + \sigma_y^{(j,i)} dW_t^3 \tag{7.3}$$

$$d\ln f_{x\,t}^{(j,i)} = \left(c_t^{(j)} - c_t^{(i)} + y_t^{(j,i)} - \frac{1}{2}(\sigma_x^{(j,i)})^2\right) dt + \sigma_x^{(j,i)} dW_t^4 \tag{7.4}$$

where $\{W^i, i = 1\cdots 4\}$ are Brownian motions under the spot martingale measure of currency $(j)$. Every $\theta(t)$ is a deterministic function of time, and is adjusted in such a way that we can recover the initial term structures of the relevant variable. We assume every $\kappa$ and $\sigma$ are constants. We allow general correlation structure ($d[W^i, W^j]_t = \rho_{i,j}dt$) except that $\rho_{3,j} = 0$ for all $j \neq 3$. The above dynamics is chosen just for simplicity and demonstrative purpose, and generic HJM framework can also be applied to the evaluation of Gateaux derivative. For details of more general dynamics in HJM framework, see Refs. [8, 9]. In the following, we use the term structure for the $(i,j)$ pair taken from the typical data of (USD, JPY) in early 2010 for presentation. In Appendix E, we have provided the term structures and other parameters used in calculation.

The discussed form of asymmetry is particularly interesting, since even if the relevant CSA is actually symmetric, the asymmetry arises effectively if there is difference in the level of sophistication of collateral management. From the following two examples, one can see that the efficient collateral management is practically relevant and the firms who are incapable of doing so will have to pay quite expensive cost to the counter party, and vice versa.

## 7.1 Asymmetric Collateralization for MtMCCOIS

We now implement Gateaux derivative using Monte Carlo simulation based on the model we have just explained. To see the reliability of Gateaux derivative, we have compared it with a numerical result directly obtained from PDE using a simplified setup in Appendix D.

Firstly, we consider MtMCCOIS explained in Sec. 6.4. We consider a spot-start, $T_N$-maturing $(i,j)$-MtMCCOIS, where the leg of currency $(i)$ (intended to be USD) needs notional refreshments. Let us assume perfect but asymmetric collateralization as follows:
(1) Party 1 is the basis spread payer and can use either the currency $(i)$ or $(j)$ as collateral.
(2) Party 2 is the basis spread receiver and can only use the currency $(i)$ as collateral.

In this case, the price of the contract at time 0 from the view point of party 1 is given by

$$V_0 = E^{Q^{(j)}}\left[\int_{]0,T_N]} \exp\left(-\int_0^s R(u, V_u)du\right) dD_s\right] \tag{7.5}$$

where

$$R(t, V_t) = c_t^{(j)} + y_t^{(j,i)} + \max(-y_t^{(j,i)}, 0)\mathbf{1}_{\{V_t<0\}} , \tag{7.6}$$

and

$$dD_s = \sum_{n=1}^{N}\left\{\delta_{T_n}(s)\left[-e^{\int_{T_{n-1}}^{T_n} c_u^{(j)}du} - \delta_n B + \frac{f_x^{(j,i)}(T_n)}{f_x^{(j,i)}(T_{n-1})}e^{\int_{T_{n-1}}^{T_n} c_u^{(i)}du}\right]\right\} . \tag{7.7}$$

Using Gateaux derivative, we can approximate the contract price as

$$V_0 \simeq V_0(0) + \nabla V_0\left(0; \max(-y^{(j,i)}, 0)\right) , \tag{7.8}$$

where

$$\begin{aligned}&\nabla V_0\left(0; \max(-y^{(j,i)}, 0)\right)\\ &= E^{Q^{(j)}}\left[\int_0^{T_N} e^{-\int_0^s (c_u^{(j)}+y_u^{(j,i)})du} \max(-V_s(0), 0) \max(-y_s^{(j,i)}, 0)ds\right] .\end{aligned} \tag{7.9}$$

Although $V_t(0)$ is simply a price under symmetric collateralization using currency $(i)$, we need to be careful about the advance reset conventions. One can show that

$$\begin{aligned} V_t(0) &= \sum_{n=\gamma(t)+1}^{N} E^{Q^{(j)}}\left[ e^{-\int_t^{T_n}(c_u^{(j)}+y_u^{(j,i)})du}\left\{ -e^{\int_{T_{n-1}}^{T_n} c_u^{(j)}du} - \delta_n B + \frac{f_x^{(j,i)}(T_n)}{f_x^{(j,i)}(T_{n-1})} e^{\int_{T_{n-1}}^{T_n} c_u^{(i)}du} \right\} \middle| \mathcal{F}_t \right] \\ &+ E^{Q^{(j)}}\left[ e^{-\int_t^{T_{\gamma(t)}}(c_u^{(j)}+y_u^{(j,i)})du}\left\{ -e^{\left(\int_{T_{\gamma(t)-1}}^{t} + \int_t^{T_{\gamma(t)}}\right)c_u^{(j)}du} - \delta_{\gamma(t)}B + \frac{e^{\int_{T_{\gamma(t)-1}}^{t} c_u^{(i)}du}}{f_x^{(j,i)}(T_{\gamma(t)-1})} e^{\int_t^{T_{\gamma(t)}} c_u^{(i)}du} f_x^{(j,i)}(T_{\gamma(t)}) \right\} \middle| \mathcal{F}_t \right], \end{aligned} \tag{7.10}$$

where $\gamma(t) = \min\{n; T_n > t, n = 1 \cdots N\}$. Note that $T_{\gamma(t)-1} < t$ since we are considering spot-start swap (or $T_0 = 0$). Assuming the independence of $y^{(j,i)}$ and other variables, we can simplify $V_t(0)$ and obtains

$$\begin{aligned} V_t(0) &= -\sum_{n=\gamma(t)}^{N} D^{(j)}(t,T_n)Y^{(j,i)}(t,T_n)\delta_n B + \sum_{n=\gamma(t)+1}^{N} D^{(j)}(t,T_{n-1})\left(Y^{(j,i)}(t,T_{n-1}) - Y^{(j,i)}(t,T_n)\right) \\ &\quad -Y^{(j,i)}(t,T_{\gamma(t)})e^{\int_{T_{\gamma(t)-1}}^{t} c_s^{(j)}ds} + \frac{f_x^{(j,i)}(t)}{f_x^{(j,i)}(T_{\gamma(t)-1})} e^{\int_{T_{\gamma(t)-1}}^{t} c_s^{(i)}ds}, \end{aligned} \tag{7.11}$$

where we have defined $Y^{(j,i)}(t,T) = E^{Q^{(j)}}\left[ e^{-\int_t^T y_s^{(j,i)}ds} \middle| \mathcal{F}_t \right]$.

In Figs. 1 and 2, we have shown the numerical result of Gateaux derivative, which is the price difference from the symmetric limit, for 10y and 20y MtMCCOIS, respectively. The spread $B$ was chosen in such way that the value in symmetric limit, $V_0(0)$, becomes zero. In both cases, the horizontal axis is the annualized volatility of $y^{(j,i)}$, and the vertical one is the price difference in terms of bps of notional of currency $(j)$. When the party 1 is the spread receiver, we have used the right axis. The results are rather insensitive to the FX volatility due to the notional refreshments of currency-$(i)$ leg. From the historical analysis performed in Ref. [10], we know that annualized volatility of $y^{(j,i)}$ tends to be 50bps or so in a calm market, but it can be $(100 \sim 200)$bps or more in a volatile market for major currency pairs, such as (EUR,USD) and (USD, JPY). Therefore, the impact of asymmetric collateralization in this example can be practically very significant when party 1 is the spread payer. When the party 1 is the spread receiver, one sees that the impact of asymmetry is very small, only a few bps of notional. This can be easily understood in the following way: When the party 1 has a negative mark-to-market value and has the option to change the collateral currency, $y^{(j,i)}$ tends to be large and hence the optimal currency remains the same currency $(i)$.

Finally, let us briefly mention about the standard MtMCCS with Libor payments. As discussed in Ref. [10], the contribution from Libor-OIS spread to CCS is not significant relative to that of $y^{(j,i)}$. Therefore, the numerical significance of asymmetric collateralization is expected to be quite similar in the standard case, too.

### 7.2 Asymmetric Collateralization for OIS

Now we study the impact of asymmetric collateralization on OIS. We consider OIS of currency $(j)$, and assume the following asymmetry in collateralization:

(1) Party 1 is the fixed receiver and can use either the currency $(i)$ or $(j)$ as collateral.
(2) Party 2 is the fixed payer can only use the currency $(j)$ (domestic currency) as collateral.

For spot-start, $T_N$-maturing OIS, we have

$$V_0 = E^{Q^{(j)}}\left[\int_{]0,T_N]} e^{-\int_0^s R(u,V_u)du} dD_s\right] , \tag{7.12}$$

where

$$dD_s = \sum_{n=1}^{N} \delta_{T_n}(s)\left[\delta_n S - \left(e^{\int_{T_{n-1}}^{T_n} c_u^{(j)} du} - 1\right)\right] , \tag{7.13}$$

and

$$R(t, V_t) = c_t^{(j)} + \max(y_t^{(j,i)}, 0)\mathbf{1}_{\{V_t<0\}} . \tag{7.14}$$

Using Gateaux derivative, the above swap value can be approximated as

$$V_0 \simeq V_0(0) + \nabla V_t\left(0; \max(y_t^{(j,i)}, 0)\right) , \tag{7.15}$$

where

$$\nabla V_0\left(0; \max(y^{(j,i)}, 0)\right) = E^{Q^{(j)}}\left[\int_0^T e^{-\int_0^s c_u^{(j)} du} \max(-V_s(0), 0) \max(y_s^{(j,i)}, 0)\, ds\right] , \tag{7.16}$$

and

$$\begin{aligned} V_t(0) &= E^{Q^{(j)}}\left[\sum_{n=\gamma(t)}^{N} e^{-\int_t^{T_n} c_u^{(j)} du}\left\{\delta_n S - \left(e^{\int_{T_{n-1}}^{T_n} c_u^{(j)} du} - 1\right)\right\}\middle| \mathcal{F}_t\right] \\ &= \sum_{n=\gamma(t)}^{N} D^{(j)}(t, T_n)\delta_n S - e^{\int_{T_{\gamma(t)-1}}^{t} c_u^{(j)} du} + D^{(j)}(t, T_N) . \end{aligned} \tag{7.17}$$

Here, $S$ is the fixed OIS rate.

In Figs. 3, 4, and 5, we have shown the numerical results Gateaux derivative for 10y and 20y OIS. In the first two figures, we have fixed $\sigma_c^{(j)} = 1\%$ and changed $\sigma_y^{(j,i)}$ to see the sensitivity against CCS. In the last figure, we have fixed the $y^{(j,i)}$ volatility as $\sigma_y^{(j,i)} = 0.75\%$ and changed the volatility of collateral rate $c^{(j)}$. Since the term structure of OIS rate is upward sloping, the mark-to-market value of a receiver tends to be negative in the long end of the contract, which makes the optionality of collateral currency choice larger and hence bigger price difference relative to the payer case.

# 8 General Implications of Asymmetric Collateralization

From the results of section 7, we have seen the practical significance of asymmetric collateralization. It is now clear that sophisticated financial firms may obtain significant funding benefit from the less-sophisticated counter parties by carrying out clever collateral strategies.

Before concluding the paper, let us explain two generic implications of collateralization one for netting and the other for resolution of information, which is closely related to the observation just explained. Although derivation itself can be done in exactly the same way as Ref. [3] after the reinterpretation of several variables, we get new insights for collateralization that can be important for the appropriate design and regulations for the financial market.

### 8.1 An implication for Netting

**Proposition 2** [5] *Assume perfect collateralization. Suppose that, for each party $i$, $y_t^i$ is bounded and does not depend on the contract value directly. Let $V^a$, $V^b$, and $V^{ab}$ be, respectively, the value processes (from the view point of party 1) of contracts with cumulative dividend processes $D^a$, $D^b$, and $D^a + D^b$. If $y^1 \geq y^2$, then $V^{ab} \geq V^a + V^b$, and if $y^1 \leq y^2$, then $V^{ab} \leq V^a + V^b$.*

Proof is available in Appendix B. The interpretation of the proposition is very clear: The party who has the higher funding cost $y$ due to asymmetric CSA or lack of sophistication in collateral management prefer to have netting agreements to decrease funding cost. On the other hand, an advanced financial firm who has capability to carry out optimal collateral strategy to achieve the lowest possible value of $y$ tries to avoid netting to exploit funding benefit. For example, an advanced firm may prefer to enter an opposite trade with a different counterparty rather than to unwind the original trade. For standardized products traded through CCPs, such a firm may prefer to use several clearing houses cleverly to avoid netting.

The above finding seems slightly worrisome for the healthy development of CCPs. Advanced financial firms that have sophisticated financial technology and operational system are usually primary members of CCPs, and some of them are trying to set up their own clearing service facility. If those firms try to exploit funding benefit, they avoid concentration of their contracts to major CCPs and may create very disperse interconnected trade networks and may reduce overall netting opportunity in the market. Although remaining credit exposure is very small as long as collateral is successfully being managed, the dispersed use of CCPs may worsen the systemic risk once it fails. In the work of Duffie & Huang [3], the corresponding proposition is derived in the context of bilateral CVA. We emphasize that one important practical difference is the strength of incentives provided to financial firms. Although it is somewhat obscure how to realize profit/loss reflected in CVA, it is rather straightforward in the case of collateralization by making use of CCS market as we have explained in the remarks of Sec. 6.4.

### 8.2 An implication for Resolution of Information

We once again follow the setup given in Ref [3]. We assume the existence of two markets: One is market $F$, which has filtration $\mathbb{F}$, that is the one we have been studying. The other one is market $G$ with filtration $\mathbb{G} = \{\mathcal{G}_t : t \in [0, T]\}$. The market $G$ is identical to the market $F$ except that it has earlier resolution of uncertainty, or in other words, $\mathcal{F}_t \subseteq \mathcal{G}_t$

[5] We assume perfect collateralization just for clearer interpretation. The results will not change qualitatively as long as $\delta^i y_t^i > (1 - R_t^i)(1 - \delta_t^i)^+ h_t^i - (1 - R_t^j)(\delta_t^i - 1)^+ h_t^j$.

for all $t \in [0, T]$ while $\mathcal{F}_0 = \mathcal{G}_0$. The spot marting measure $Q$ is assumed to apply to the both markets.

**Proposition 3** [6] *Assume perfect collateralization. Suppose that, for each party $i$, $y^i$ is bounded and does not depend on the contract value directly. Suppose that $r$, $y^1$ and $y^2$ are adapted to both the filtrations $\mathbb{F}$ and $\mathbb{G}$. The contract has cumulative dividend process $D$, which is a semimartingale of integrable variation with respect to filtrations $\mathbb{F}$ and $\mathbb{G}$. Let $V^F$ and $V^G$ denote, respectively, the values of the contract in markets $F$ and $G$ from the view point of party 1. If $y^1 \geq y^2$, then $V_0^F \geq V_0^G$, and if $y^1 \leq y^2$, then $V_0^F \leq V_0^G$.*

Proof is available in Appendix C. The proposition implies that the party who has the higher effective funding cost $y$ either from the lack of sophisticated collateral management technique or from asymmetric CSA would like to delay the information resolution to avoid timely margin call from the counterparty. The opposite is true for advanced financial firms which are likely to have advantageous CSA and sophisticated system. The incentives to obtain funding benefit will urge these firms to provide mark-to-market information of contracts to counter parties in timely manner, and seek early resolution of valuation dispute to achieve significant funding benefit. Considering the privileged status of these firms, the latter effects will probably be dominant in the market.

# 9 Conclusions

This article develops the methodology to deal with asymmetric and imperfect collateralization as well as remaining counterparty credit risk. It was shown that all of the issues are able to be handled in an unified way by making use of Gateaux derivative. We have shown that the resulting formula contains CCA that represents adjustment of collateral cost due to the deviation from the perfect collateralization, and the terms corresponding CVA, which now contains the possible dependency among cost of collaterals, hazard rates, collateral coverage ratio and the underlying contract value. Even if we assume that the collateral coverage ratio and recovery rate are constant, the change of effective discounting rate induced by collateral cost and its correlation to other variables may significantly change the value of CVA.

Direct link of CCS spread and collateral cost allows us to study the numerical significance of asymmetric collateralization. From the numerical analysis using CCS and OIS, the relevance of sophisticated collateral management is now clear. If a financial firm is incapable of choosing the cheapest collateral currency, it has to pay very expensive funding cost to the counter party. We also explained the issue of one-way CSA, which is common when SSA entities are involved. If the funding cost of collateral (or "$y$") rises, the financial firm that is the counterparty of SSA may suffer from significant loss of mark-to-market value as well as the huge cash-flow mismatch.

The article also discussed some generic implications of collateralization. In particular, it was shown that the sophisticated financial firms are likely to avoid netting of trades if they try to exploit funding benefit as much as possible, which may reduce the overall netting opportunity and potentially increase the systemic risk in the financial market.

[6] We assume perfect collateralization just for clearer interpretation. The results will not change qualitatively as long as $\delta^i y_t^i > (1 - R_t^i)(1 - \delta_t^i)^+ h_t^i - (1 - R_t^j)(\delta_t^i - 1)^+ h_t^j$.

# A Proof of Proposition 1

Firstly, we consider the SDE for $S_t$. Let us define $L_t = 1 - H_t$. One can show that

$$\beta_t^{-1} S_t + \int_{]0,t]} \beta_u^{-1} L_u \big(dD_u + q(u, S_u) S_u du\big) + \int_{]0,t]} \beta_u^{-1} L_{u-} \big(Z^1(u, S_{u-}) dH_u^1 + Z^2(u, S_{u-}) dH_u^2\big)$$

$$= \mathbf{E}^Q \left[ \int_{]0,T]} \beta_u^{-1} \mathbf{1}_{\{\tau > u\}} \Big\{ dD_u + \big(y_u^1 \delta_u^1 \mathbf{1}_{\{S_u < 0\}} + y_u^2 \delta_u^2 \mathbf{1}_{\{S_u \geq 0\}}\big) S_u du \Big\} \right.$$

$$\left. + \int_{]0,T]} \beta_u^{-1} L_{u-} \Big( Z^1(u, S_{u-}) dH_u^1 + Z^2(u, S_{u-}) dH_u^2 \Big) \middle| \mathcal{F}_t \right] = m_t \qquad \text{(A.1)}$$

where

$$q(t, v) = y_t^1 \delta_t^1 \mathbf{1}_{\{v < 0\}} + y_t^2 \delta_t^2 \mathbf{1}_{\{v \geq 0\}} \qquad \text{(A.2)}$$

and $\{m_t\}_{t \geq 0}$ is a $Q$-martingale. Thus we obtain the following SDE:

$$dS_t - r_t S_t dt + L_t \big(dD_t + q(t, S_t) S_t dt\big) + L_{t-} \big(Z^1(t, S_{t-}) dH_t^1 + Z^2(t, S_{t-}) dH_t^2\big) = \beta_t dm_t \ . \qquad \text{(A.3)}$$

Using the decomposition of $H_t^i$, we get

$$dS_t - r_t S_t dt + L_t \big(dD_t + q(t, S_t) S_t dt\big) + L_t \big(Z^1(t, S_t) h_t^1 + Z^2(t, S_t) h_t^2\big) dt = dn_t \ , \qquad \text{(A.4)}$$

where we have defined

$$dn_t = \beta_t dm_t - L_{t-} \big(Z^1(t, S_{t-}) dM_t^1 + Z^2(t, S_{t-}) dM_t^2\big) \qquad \text{(A.5)}$$

and $\{n_t\}_{t \geq 0}$ is also a some $Q$-martingale. Using the fact that

$$q(t, S_t) S_t + Z^1(t, S_t) h_t^1 + Z^2(t, S_t) h_t^2 = S_t \big(\mu(t, S_t) + h_t\big) \ , \qquad \text{(A.6)}$$

one can show that the SDE for $S_t$ is given by

$$dS_t = -L_t dD_t + L_t \big(r_t - \mu(t, S_t) - h_t\big) S_t dt + dn_t \ . \qquad \text{(A.7)}$$

Secondly, let us consider the SDE for $V_t$. By following the similar procedures, one can easily see that

$$e^{-\int_0^t (r_u - \mu(u, V_u)) du} V_t + \int_{]0,t]} e^{-\int_0^s (r_u - \mu(u, V_u)) du} dD_s$$

$$= E^Q \left[ \int_{]0,T]} \exp\left( - \int_0^s \big(r_u - \mu(u, V_u)\big) du \right) dD_u \middle| \mathcal{F}_t \right] = \tilde{m}_t \ , \qquad \text{(A.8)}$$

where $\{\tilde{m}_t\}_{t \geq 0}$ is a $Q$-martingale. Thus we have

$$dV_t = -dD_t + \big(r_t - \mu(t, V_t)\big) V_t dt + d\tilde{n}_t \ , \qquad \text{(A.9)}$$

where

$$d\tilde{n}_t = e^{\int_0^t (r_u - \mu(u, V_u)) du} d\tilde{m}_t \ , \qquad \text{(A.10)}$$

and hence $\{\tilde{n}_t\}_{t\geq 0}$ is also a Q-martingale. As a result we have

$$\begin{aligned}
d(\mathbf{1}_{\{\tau>t\}}V_t) &= d(L_tV_t) \\
&= L_{t-}dV_t - V_{t-}dH_t - \Delta V_\tau \Delta H_\tau \\
&= -L_{t-}dD_t + L_t\big(r_t - \mu(t,V_t)\big)V_t dt - L_t V_t h_t dt - \Delta V_\tau \Delta H_\tau \\
&\quad + L_{t-}\left(d\tilde{n}_t - V_{t-}(dM_t^1 + dM_t^2)\right) \\
&= -L_t dD_t + L_t\big(r_t - \mu(t,V_t) - h_t\big)V_t dt - \Delta V_\tau \Delta H_\tau + d\tilde{N}_t, \qquad \text{(A.11)}
\end{aligned}$$

where $\{\tilde{N}_t\}_{t\geq 0}$ is a Q-martingale such that

$$d\tilde{N}_t = L_{t-}\left(d\tilde{n}_t - V_{t-}(dM_t^1 + dM_t^2)\right) . \qquad \text{(A.12)}$$

Therefore, by comparing Eqs. (A.7) and (A.11) and also the fact that $S_T = \mathbf{1}_{\{\tau>T\}}V_T = 0$, we cannot distinguish $\mathbf{1}_{\{\tau>t\}}V_t$ from $S_t$ if there is no jump at the time of default $\Delta V_\tau = 0$. ■

## B Proof of Proposition 2

Consider the case of $y^1 \geq y^2$. From Eq. (2.6), one can show that the pre-default value $V$ can also be written in the following recursive form:

$$V_t = E^Q\left[-\int_{]t,T]}\big(r_s - \mu(s,V_s)\big)V_s ds + \int_{]t,T]} dD_s \,\middle|\, \mathcal{F}_t\right] . \qquad \text{(B.1)}$$

Let us define the following variables:

$$\begin{aligned}
\tilde{V}_t &= e^{-\int_0^t (r_s - y_s^1)ds} V_t \qquad &\text{(B.2)}\\
\tilde{D}_t &= \int_{]0,t]} e^{-\int_0^s (r_u - y_u^1)du} dD_s . \qquad &\text{(B.3)}
\end{aligned}$$

Note that

$$\begin{aligned}
r_t - \mu(t,V_t) &= (r_t - y_t^1) + (y_t^1 - y_t^2)\mathbf{1}_{\{V_t\geq 0\}} \\
&= (r_t - y_t^1) + \eta_t^{1,2}\mathbf{1}_{\{V_t\geq 0\}} , \qquad \text{(B.4)}
\end{aligned}$$

where we have defined $\eta^{i,j} = y^i - y^j$. Using new variables, Eq. (B.1) can be rewritten as

$$\tilde{V}_t = E^Q\left[-\int_{]t,T]} \eta_s^{1,2}\mathbf{1}_{\{\tilde{V}_s\geq 0\}}\tilde{V}_s ds + \int_{]t,T]} d\tilde{D}_s \,\middle|\, \mathcal{F}_t\right] . \qquad \text{(B.5)}$$

And hence we have,

$$\tilde{V}_t^{ab} - \tilde{V}_t^a - \tilde{V}_t^b = E^Q\left[-\int_{]t,T]} \eta_s^{1,2}\left(\max(\tilde{V}_s^{ab},0) - \max(\tilde{V}_s^a,0) - \max(\tilde{V}_s^b,0)\right) ds \,\middle|\, \mathcal{F}_t\right] . \qquad \text{(B.6)}$$

Let us denote the upper bound of $\eta^{1,2}$ as $\alpha$, and also define $Y = \tilde{V}^{ab} - \tilde{V}^{a} - \tilde{V}^{b}$ and $G_s = -\eta_s^{1,2}\left(\max(\tilde{V}_s^{ab},0) - \max(\tilde{V}_s^{a},0) - \max(\tilde{V}_s^{b},0)\right)$. Then, we have $Y_T = 0$ and

$$Y = E^Q\left[\int_{]t,T]} G_s ds \middle| \mathcal{F}_t\right] . \tag{B.7}$$

$$\begin{aligned} G_s &= -\eta_s^{1,2}\left(\max(\tilde{V}_s^{ab},0) - \max(\tilde{V}_s^{a},0) - \max(\tilde{V}_s^{b},0)\right) \\ &\geq -\eta_s^{1,2}\left(\max(\tilde{V}_s^{ab},0) - \max(\tilde{V}_s^{a} + \tilde{V}_s^{b},0)\right) \\ &\geq -\eta_s^{1,2}\max\left(\tilde{V}_s^{ab} - \tilde{V}_s^{a} - \tilde{V}_s^{b},0\right) \\ &\geq -\alpha|Y_s| . \end{aligned} \tag{B.8}$$

Applying the consequence of the Stochastic Gronwall-Bellman Inequality in Lemma **B2** of Ref. [4] to $Y$ and $G$, we can conclude $Y_t \geq 0$ for all $t \in [0,T]$, and hence $V^{ab} \geq V^a + V^b$.■

# C Proof of Proposition 3

Consider the case of $y^1 \geq y^2$. Let us define

$$\tilde{V}_t^F = e^{-\int_0^t (r_s - y_s^1)ds} V_t^F \tag{C.1}$$

$$\tilde{V}_t^G = e^{-\int_0^t (r_s - y_s^1)ds} V_t^G , \tag{C.2}$$

as well as

$$\tilde{D}_t = \int_{]0,t]} e^{-\int_0^s (r_u - y_u^1)du} dD_s \tag{C.3}$$

as in the previous section. Then, we have

$$\tilde{V}_t^G = E^Q\left[-\int_{]t,T]} \eta_s^{1,2}\max(\tilde{V}_s^G,0)ds + \int_{]t,T]} d\tilde{D}_s \middle| \mathcal{G}_t\right] \tag{C.4}$$

$$\tilde{V}_t^F = E^Q\left[-\int_{]t,T]} \eta_s^{1,2}\max(\tilde{V}_s^F,0)ds + \int_{]t,T]} d\tilde{D}_s \middle| \mathcal{F}_t\right] . \tag{C.5}$$

Now, let us define

$$U_t = E^Q\left[\tilde{V}_t^G \middle| \mathcal{F}_t\right] . \tag{C.6}$$

Then, using Jensen's inequality, we have

$$U_t \leq E^Q\left[-\int_{]t,T]} \eta_s^{1,2}\max(U_s,0)ds + \int_{]t,T]} d\tilde{D}_s \middle| \mathcal{F}_t\right] . \tag{C.7}$$

Therefore, we obtain

$$\tilde{V}_t^F - U_t \geq E^Q\left[-\int_{]t,T]} \eta_s^{1,2}\left(\max(\tilde{V}_s^F,0) - \max(U_s,0)\right) ds \middle| \mathcal{F}_t\right] \tag{C.8}$$

$$\geq E^Q\left[-\int_{]t,T]} \eta_s^{1,2}|\tilde{V}_s^F - U_s| ds \middle| \mathcal{F}_t\right] . \tag{C.9}$$

Using the stochastic Gronwall-Bellman Inequality as before, one can conclude that $\tilde{V}_t^F \geq U_t$ for all $t \in [0,T]$, and in particular, $V_0^F \geq V_0^G$.■

# D Comparison of Gateaux Derivative with PDE

In order to get clear image for the reliability of Gateaux derivative, we compare it with the numerical result directly obtained from PDE. We consider a simplified setup where MtM-CCOIS exchanges the coupons continuously, and the only stochastic variable is a spread $y$. Consider continuous payment $(i,j)$-MtMCCOIS where the leg of currency $(i)$ needs notional refreshments. We assume following situation as the asymmetric collateralization: (1) Party 1 is the basis spread payer and can use either the currency $(i)$ or $(j)$ as collateral. (2) Party 2 is the basis spread receiver and can only use the currency $(i)$ as collateral.

In this case, one can see that the value of $t$-start $T$-maturing contract from the view point of party 1 is given by (See, Eq. (6.19).)

$$V_t = E^{Q^{(j)}}\left[\int_t^T \exp\left(-\int_t^s R(u,V_u)du\right)\left(y_s^{(j,i)} - B\right)ds \middle| \mathcal{F}_t\right] , \tag{D.1}$$

where

$$R(t,V_t) = c^{(j)}(t) + y_t^{(j,i)} + \max\left(-y_t^{(j,i)}, 0\right)\mathbf{1}_{\{V_t<0\}} \tag{D.2}$$

and $B$ is a fixed spread for the contract. $y^{(j,i)}$ is the only stochastic variable and its dynamics is assumed to be given by the following Hull-White model:

$$dy_t^{(j,i)} = \left(\theta^{(j,i)}(t) - \kappa^{(j,i)}y_t^{(j,i)}\right)dt + \sigma_y^{(j,i)}dW_t^{Q^{(j)}} . \tag{D.3}$$

Here, $\theta^{(j,i)}(t)$ is a deterministic function specified by the initial term structure of $y^{(j,i)}$, $\kappa^{(j,i)}$ and $\sigma_y^{(j,i)}$ are constants. $W^{Q^{(j)}}$ is a Brownian motion under the spot martingale measure of currency $(j)$.

The PDE for $V_t$ is given by

$$\frac{\partial}{\partial t}V(t,y) + \left(\gamma(t,y)\frac{\partial V(t,y)}{\partial y} + \frac{(\sigma_y^{(j,i)})^2}{2}\frac{\partial^2}{\partial y^2}V(t,y)\right) - R(t,V(t,y))V(t,y) + y - B = 0 , \tag{D.4}$$

where

$$\gamma(t,y) = \theta^{(j,i)}(t) - \kappa^{(j,i)}y . \tag{D.5}$$

If party 1 is a spread receiver, we need to change $y - B$ to $B - y$, of course.

Terminal boundary condition is trivially given by $V(T,\cdot) = 0$. On the lower boundary of $y$ or when $y = -M$ $(= y_{\min}) \ll 0$, we have $V_t < 0$ for all $t$. Thus, we have $R(s,V(s,y)) = c^{(j)}(s)$ for all $s \geq t$, if $y = -M$ at time $t$. Therefore, on the lower boundary, the value of MtMCCOIS is given by

$$\begin{aligned} V(t,-M) &= E^{Q^{(j)}}\left[\int_t^T e^{-\int_t^s c_u^{(j)}du}(y_s^{(j,i)} - B)ds \middle| y_t^{(j,i)} = -M\right] \\ &= \int_t^T D^{(j)}(t,s)\left(-B - \frac{\partial}{\partial s}\ln Y^{(j,i)}(t,s)\right)ds . \end{aligned} \tag{D.6}$$

Since $c^{(j)}(t)$ is a deterministic function, $D^{(j)}(t,s) = D^{(j)}(0,s)/D^{(j)}(0,t)$ is simply given by the forward.

On the other hand, when $y = M\ (= y_{\max}) \gg 0$, we have $V_t > 0$ for all $t$. Thus we have $R(s, V(s,y)) = c^{(j)}(s) + y^{(j,i)}(s)$ for all $s \geq t$, if $y = M$ at time $t$. Thus, on the upper boundary, the value of the contract becomes

$$\begin{aligned} V(t,M) &= E^{Q^{(j)}}\left[\int_t^T e^{-\int_t^s \left(c_u^{(j)}+y_u^{(j,i)}\right)du}\left(y_s^{(j,i)} - B\right)\middle| y_t^{(j,i)} = M\right] \\ &= \int_t^T \left\{-BD^{(j)}(t,s)Y^{(j,i)}(t,s) - D^{(j)}(t,s)\frac{\partial}{\partial s}Y^{(j,i)}(t,s)\right\} ds\ . \end{aligned} \tag{D.7}$$

Now let us compare the numerical result between Gateaux derivative and PDE. In the case of Gateaux derivative, the contract value is approximated as

$$V_t \simeq V_t(0) + \nabla V_t\left(0; \max(-y^{(j,i)}, 0)\right)\ , \tag{D.8}$$

where

$$V_t(0) = E^{Q^{(j)}}\left[\int_t^T e^{-\int_t^s (c_u^{(j)}+y_u^{(j,i)})du}\left(y_s^{(j,i)} - B\right) ds\middle| \mathcal{F}_t\right]\ , \tag{D.9}$$

and

$$\begin{aligned} &\nabla V_t\left(0; \max(-y^{(j,i)}, 0)\right) \\ &= E^{Q^{(j)}}\left[\int_t^T e^{-\int_t^s (c_u^{(j)}+y_u^{(j,i)})du} \max\left(-V_s(0), 0\right) \max\left(-y_s^{(j,i)}, 0\right) ds\middle| \mathcal{F}_t\right]. \end{aligned} \tag{D.10}$$

$V_t(0)$ is the value of the contract under symmetric collateralization where both parties post currency $(i)$ as collateral, and $\nabla V_t$ is a deviation from it.

In Fig. 6, we plot the price difference of continuous 10y-MtMCCOIS from its symmetric limit obtained by PDE and Gateaux derivative with various volatility of $y^{(j,i)}$. Term structures of $y^{(j,i)}$ and other curves are given in Appendix E. Here, the spread $B$ is chosen in such a way that the swap price is zero in the case where both parties can only use currency $(i)$ as collateral, or B is a market par spread. The price difference is $V_t - V_t(0)$ and expressed as basis points of notional. From our analysis using the recent historical data in Ref. [10], we know that the annualized volatility of $y$ is around 50 bps for a calm market but it can be more than $(100 \sim 200)$ bps when CCS market is volatile (We have used EUR/USD and USD/JPY pairs.). One observes that Gateaux derivative provides reasonable approximation for wide range of volatility. If the party 1 is a spread receiver, both of the methods give very small price differences, less than 1bp of notional.

# E Data used in Numerical Studies

The parameter we have used in simulation are

$$\kappa^{(j)} = \kappa^{(i)} = 1.5\% \tag{E.1}$$

$$\sigma_c^{(j)} = \sigma_c^{(i)} = 1\% \tag{E.2}$$

$$\sigma_x^{(j,i)} = 12\%\ . \tag{E.3}$$

All of them are defined in annualized term. The volatility of $y^{(j,i)}$ is specified in the main text in each numerical analysis.

Term structures and correlation used in simulation are given in Fig. 7. There we have defined

$$R_{\mathrm{OIS}}^{(k)}(T) = -\frac{1}{T}\ln E^{Q^{(k)}}\left[e^{-\int_0^T c_s^{(k)}ds}\right]$$
$$R_{y^{(j,i)}}(T) = -\frac{1}{T}\ln E^{Q^{(j)}}\left[e^{-\int_0^T y_s^{(j,i)}ds}\right] .$$

The curve data is based on the calibration result of typical JPY and USD market data of early 2010. In Monte Carlo simulation, in order to reduce simulation error, we have adjusted drift terms $\theta(t)$ to achieve exact match to the relevant forwards in each time step.

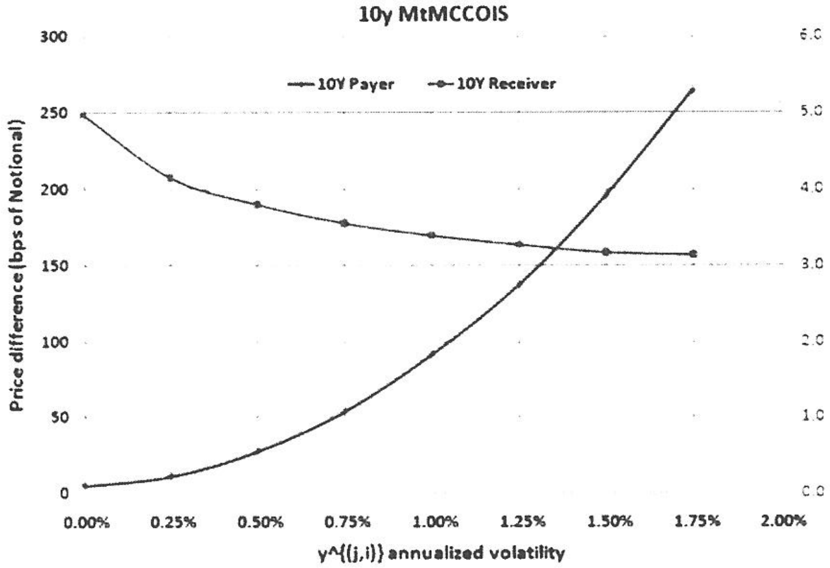


Figure 1: Price difference from symmetric limit for 10y MtMCCOIS

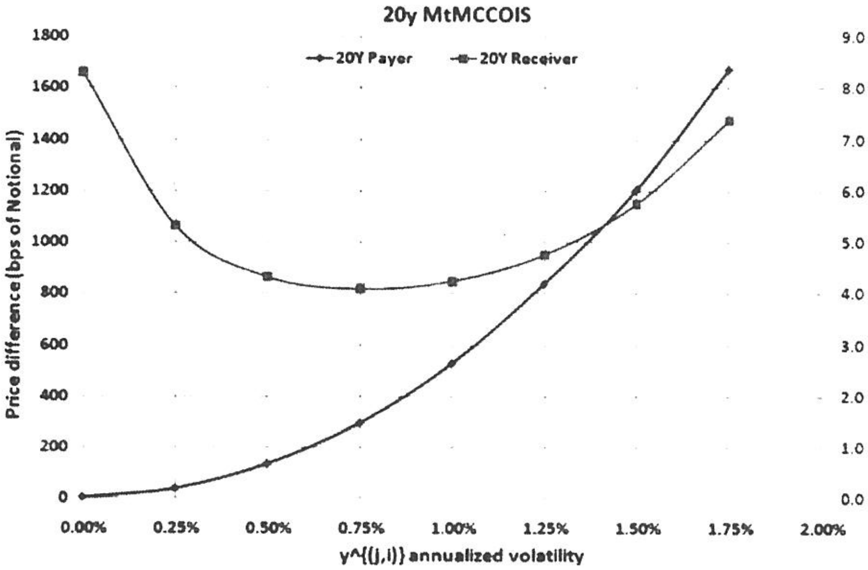


Figure 2: Price difference from symmetric limit for 20y MtMCCOIS

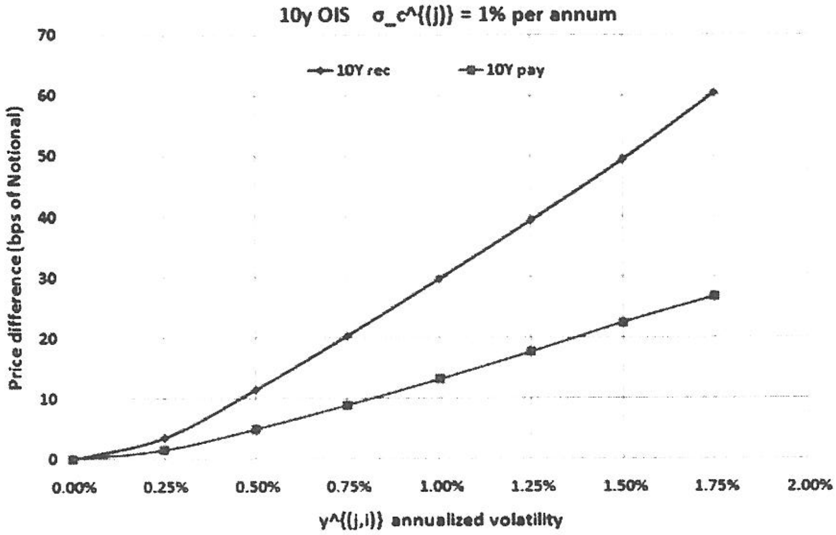


Figure 3: Price difference from symmetric limit for 10y OIS

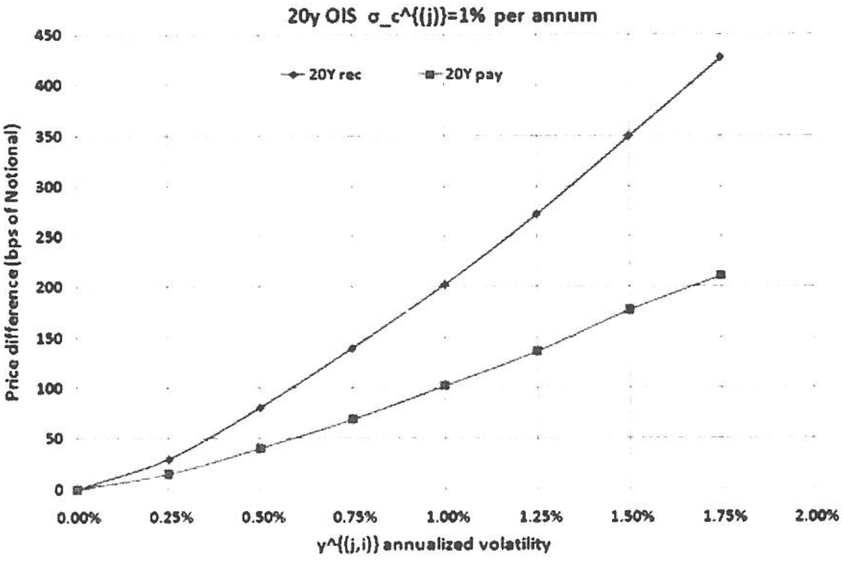


Figure 4: Price difference from symmetric limit for 20y OIS

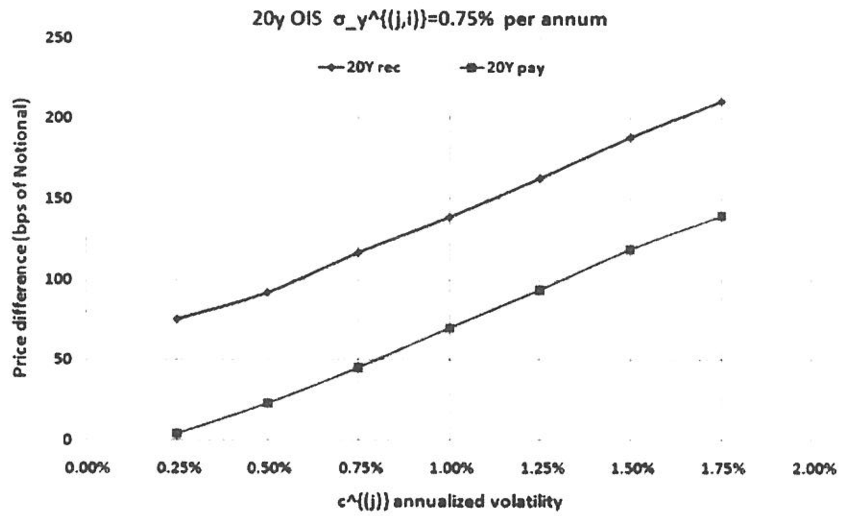


Figure 5: Price difference from symmetric limit for 20y OIS for the change of $\sigma_c^{(j)}$

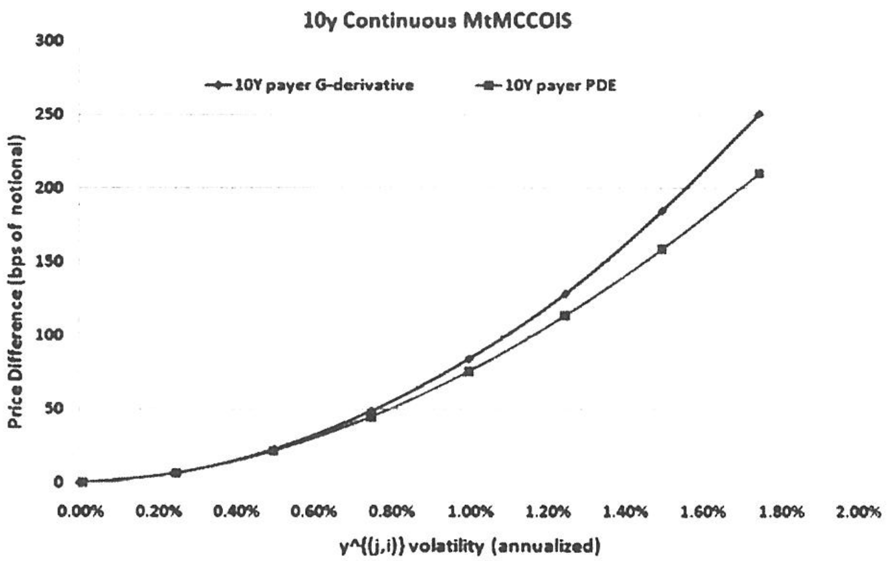


Figure 6: Price difference from symmetric limit for 10y continuous MtMCCOIS

Instantaneous correlation ρ

| | cj | cl | fx(j,i) | yji |
|---|---|---|---|---|
| cj | 100% | 24% | -5% | 0% |
| cl | 24% | 100% | 15% | 0% |
| fx | -5% | 15% | 100% | 0% |
| yji | 0% | 0% | 0% | 100% |

OIS of currency (j)

| term | R_OIS |
|---|---|
| 1D | 0.0950% |
| 1M | 0.0943% |
| 3M | 0.0943% |
| 6M | 0.0960% |
| 1Y | 0.0970% |
| 18M | 0.1060% |
| 2Y | 0.1313% |
| 3Y | 0.1866% |
| 4Y | 0.2805% |
| 5Y | 0.3883% |
| 6Y | 0.5167% |
| 7Y | 0.6584% |
| 8Y | 0.8000% |
| 9Y | 0.9418% |
| 10Y | 1.0772% |
| 12Y | 1.3163% |
| 15Y | 1.5727% |
| 20Y | 1.8339% |
| 25Y | 1.9249% |
| 30Y | 1.9627% |

OIS of currency (i)

| term | R_OIS |
|---|---|
| 0D | 0.2527% |
| 3m | 0.2547% |
| 1y | 0.3886% |
| 2y | 0.8272% |
| 3y | 1.3508% |
| 4y | 1.8071% |
| 5y | 2.1577% |
| 7y | 2.6198% |
| 10y | 3.0111% |
| 12y | 3.1560% |
| 15y | 3.2666% |
| 20y | 3.2659% |
| 25y | 3.2452% |
| 30y | 3.2224% |

y^{(j,i)} spread

| term | Ryij |
|---|---|
| 0D | -0.174% |
| 1Y | -0.197% |
| 2Y | -0.232% |
| 3Y | -0.262% |
| 4Y | -0.287% |
| 5Y | -0.301% |
| 6Y | -0.309% |
| 7Y | -0.312% |
| 8Y | -0.308% |
| 9Y | -0.307% |
| 10Y | -0.300% |
| 12Y | -0.288% |
| 15Y | -0.270% |
| 20Y | -0.242% |
| 25Y | -0.211% |
| 30Y | -0.188% |

Figure 7: Term structures and correlation used for simulation

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