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Optimal Loan Portfolio under Regulatory and Internal Constraints

Makoto Okawara
The University of Tokyo

Akihiko Takahashi
The University of Tokyo

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Makoto Okawara *Akihiko Takahashi†

Abstract

The environment surrounding banks is becoming increasingly severe. Particularly, to prevent the next financial crisis, Basel III requires financial institutions to prepare higher levels of capitals by January 1st, 2028, and the financial stability board (FSB) suggests the risk appetite framework (RAF) as their internal risk management. Hence, efficient usage of their own capitals for banks is more important than ever to improve profitability. Under such circumstances, this paper is the first to consider an optimization problem for a typical loan portfolio of international banks under comprehensive risk constraints with realistic profit margins and funding costs to achieve an efficient capital allocation. Concretely, after taking concentration risks on large individual obligors into account, we obtain a loan portfolio that attains the maximum profit under Basel regulatory capital and loan market constraints, as well as internal management constraints, namely risk limits on business units and industrial sectors. Moreover, we separately calculate credit risk amounts of the internal constraints in terms of regulatory and economic capitals to compare the optimized profits. In addition, considering sharp increases in default probabilities of all obligors as in the global financial crisis, we perform a stress test on the optimization results to investigate the effects of changes in risk amounts and profits. As a result, we propose to unify risk constraints on the business units and industrial sectors by using credit risk amounts in terms of economic capitals.

Keywords: regulatory capital; economic capital; risk appetite framework(RAF); granularity adjustment; constrained optimization

1 Introduction

1.1 Introduction

The business environment of financial institutions is becoming increasingly severe. First, the COVID-19 epidemic and the geopolitical risks have increased uncertainty about the future of the global economy, and there is a possibility that local credit deterioration will spread internationally. Second, financial institutions are forced to review their loan portfolios to promote the realization of a de-carbonized society due to the growing impact of global warming on the environment and society. Third, to prevent a recurrence of the global financial crisis and stabilize the financial system, Basel III will significantly raise the capital requirement for financial institutions by January 1st, 2028. In addition, the financial stability board (FSB) suggests the

* Graduate School of Economics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo, 113-0033, Japan, okawaramakoto@gmail.com

† Graduate School of Economics, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo, 113-0033, Japan, akihikot@e.u-tokyo.ac.jp

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risk appetite framework (hereafter RAF) introduced in [7] for financial institutions to prevent the next financial crisis. Since those aim to strengthen risk management rather than earn profits through business expansion, efficient capital utilization is more important than ever to improve profitability and soundness in bank management, where capital utilization is defined as "given funds held by a financial institution, taking risks efficiently and aiming to improve profitability from a company-wide perspective". Therefore, this paper considers capital allocation that achieves maximum profit under various realistic constraints.

Next, let us briefly explain major constraints international banks face to construct a loan portfolio in practice. Firstly, they should take the capital constraint under the Basel regulation into account. It is the minimum capital requirement imposed by financial authorities on banks. Secondly, they need to consider risk constraints based on RAF as their internal management, such as risk appetite on each business unit and risk limit on each industrial segment to achieve their management goals. Finally, they must include some constraint in the loan market arising from lack of demand or/and competitiveness of loans in financing markets.

Taking those constraints into account, this paper investigates an optimization problem for a typical loan portfolio of international banks. More concretely, considering concentration risks on large individual obligors within each industrial sector, we obtain a capital allocation to lending, which achieves the maximum profit under the Basel regulatory capital and loan market constraints, as well as constraints by internal management, namely risk limits on each business unit (hereafter BU) and industrial sector.

Moreover, preparing a portfolio with the concentration of loans on a large individual obligor, we separately calculate the internal constraints' credit risk amounts in terms of regulatory and economic capitals to compare the optimized profits, where the regulatory capital is defined as "the minimum amount of funds required by financial authorities to ensure soundness of the entire financial system," and the economic capital is defined as "the funds that financial institutions should prepare for potential losses arising from the risks associated with their businesses".

In addition, by using granularity adjustment to incorporate concentration risks on large individual obligors and setting realistic profit margins with funding costs, we optimize a loan portfolio under the constraints and risk limits mentioned above. Furthermore, taking sharp increases in default probabilities of all obligors as in the global financial crisis into account, we perform stress tests on the optimization results to compare the effects of changes in risk amounts and profits.

In this way, the current paper solves an optimization problem of a loan portfolio more realistically and comprehensively than existing research, and analyzes the effects of different risk constraints on profits from multiple perspectives. As a result, we propose to unify internal risk constraints on the business units and industrial sectors by using credit risk amounts in terms of economic capitals.

The organization of this paper is as follows: The next subsection discusses related previous works. Section 2 shows an approximate expression for credit risk, applies it to a one-factor Merton model, and introduces a regulatory and economic capital credit risks. Then, we formulate an optimization problem of a loan portfolio under constraints. Section 3 estimates optimal portfolios and examines the result in terms of risk and profit amounts. In addition, we present an example for a stress test. Section 4 concludes. Appendix shows details of industrial sectors and the exposure allocation into each BU (business unit).

1.2 Previous research

Previous works on loan portfolio optimization include Goto et al.[9]. They divide a loan portfolio into segments by industrial sector and rating to apply factor analysis to the TSE (Tokyo Stock Exchange) industrial sector-specific stock price index data and use the results to calculate a corporate value by using a multi-factor Merton model. Moreover, as constraints, they set (a) a minimum interest rate, (b) a maximum exposure ratio per industrial sector, and (c) a maximum exposure ratio per obligor. Then, they seek an optimal portfolio by minimizing CVaR (Conditional Value at Risk) rather than profit optimization. In addition, as a sensitivity analysis of an optimal portfolio different from our stress test, they show an optimal exposure ratio per lender when changing the number of common factors and the parameter (factor loading) that determines the ratio of the influence of common factors and individual factors. Based on this result, they point out a possibility that high factor loading, as indicated by the Basel II, would motivate lending to high credit ratings. They also show that risk limit constraints (b) and (c) provide a certain brake on this trend. However, since the credit exposure within each segment is assumed to be equal in a portfolio, concentration of credit risk on large individual obligors is not taken into account. Moreover, they do not consider risk constraints based on RAF as bank's internal management, nor investigate important issues such as capital allocation to business units, and do not examine the tightening impact of regulatory constraints on the optimal portfolio, either.

The next previous research discussed is Grzegorz[10], where risk amount is calculated in terms of the regulatory capital for a portfolio of European banks' assets and liabilities classified by country. Moreover, he sets constraints on (a) total capital and (b) liquidity, and derives an optimal portfolio by maximizing returns after adjusting risk and cost. In addition, by sensitivity analysis and a stress test of the ECB (European Central Bank) scenario, they show that a rise in default probability and an outflow of deposits contribute to a decrease in lending, and that an increase in funding costs leads to an extension of a lending period and decrease in lending. Based on this result, he argues that it is important to actively manage the spread between loans and deposits to earn profits, which is however difficult to achieve. Although profit maximization is his objective function, the analysis targets the whole financial system, and does not consider risk limits, which are important in the internal management of financial institutions. Also, comparisons with economic capital-based optimal portfolios are not investigated.

Among other previous works, Misra, and Sebastian[13] seeks an optimal portfolio that maximizes the Sharpe ratio on an efficient frontier in a CAPM mean-variance portfolio framework, where the upper and lower limits of the loan are set for each industrial sector. However, the risk amount does not consider the concentration risk of large individual obligors. In addition, although a genetic algorithm is used for optimization, a stress test reflecting a sharp economic downturn or a financial crisis against an optimized portfolio and its impact on capital are not investigated.

Moreover, Nehrebecka[17] is an interesting related study. To obtain precise estimates of industrial sector concentration risks, she represents inter-sector correlations by a time-series model with a stock price index and computes the risk amount using a method based on a multi-factor Merton model. In addition, she compares changes in the risk amount by the industrial sector and company size before and after COVID-19 in credit portfolios and confirms that an increase and decrease of the risk amount are significantly different for each segment. Based on these results, she points out that a detailed understanding of industrial sector concentration risk is important not only for banks specializing in specific industrial sectors, but also for stabilizing the financial system. However, issues such as portfolio optimization and effective usage of capital are not considered.

2 Method

2.1 Approximate expression of credit risk

This subsection formulates the quantile of a loss rate distribution using the one-factor Merton model based on Pykhtin and Dev[18], Martin and Wilde[12], Gordy[8] and Ando[1] to obtain an approximate expression for a credit risk amount of a loan portfolio under a certain confidence level: Section 2.1.1 defines the loss rate of a portfolio, and divide it into systematic and non-systematic risks. Then, Section 2.1.2 approximately derives the quantile of the loss rate distribution by a Taylor expansion, and Section 2.1.3 applies this approximate expression to the framework of the one-factor Merton model.

2.1.1 Setting

Let $(\Omega, \mathscr{F}, P)$ be a probability space, on which all the random variables introduced in this paper are defined. Let M be a number of obligors in a loan portfolio and $A_i > 0$ be a given positive real constant that represents a loan exposure to obligor i. Hereafter, we assume that a loan loss will occur only if the obligor defaults within one year and that the loss value does not change during the period, for instance due to credit deterioration. Then, we define a random variable U_i representing a loss rate incurred by obligor i as follows:

$$U_i = \begin{cases} Q_i \in [0,1] & \text{(in case of default)} \\ 0 & \text{(in case of non-default)} \end{cases},$$

where a loss rate U_i conditioned on i's default is expressed by a random variable $Q_i \in [0,1]$, of which mean and standard deviation are denoted as μ_i and σ_i, respectively.

Let us also define a loss rate L_M of a loan portfolio as follows:

$$L_M = \sum_{i=1}^{M}(U_i A_i) / \sum_{i=1}^{M} A_i.$$

Next, we introduce an one-dimensional real-valued random variable X common among all obligors, namely a systematic risk that affects the default of all obligors. For instance, X may be regarded as a global economic condition. Then, L_M is decomposed as $L_M = E[L_M|X] + W$ with a random variable W such that $E[E[L_M|X]\ W] = 0$, where W may be interpreted as a part of the loss rate which is not explainable by the systematic risk X.

Also, we put the following condition.

Assumption 2.1. *(i) U_i $(i, \ldots, M)$ are independent conditioned on X.*

(ii) For any obligors i, a conditional expected loss rate $E[U_i|X = x]$ is a strictly decreasing function of $x \in R$.

We remark that $E[U_i|X = x] := g(x)$ with some Borel function $g : R \to R$ such that $g(X(\omega)) = E[U_i|X(\omega)]$ (e.g. Definition B.34 in Björk[6]), and Assumption 2.1-(ii) means that as x increases, that is the economy improves, a conditional expected loss rate $E[U_i|X = x]$ decreases.

2.1.2 Quantile of the loss rate distribution

Firstly, let us define an α-quantile of a loss rate L_M of a loan portfolio:

Definition 2.1. *Let $q_\alpha(L_M)$ be an α-quantile of the loss rate distribution L_M. For a variable y, the α-quantile is defined as*

$$q_\alpha(L_M) = \inf\{y : P(L_M \le y) \ge \alpha\}.$$

Then, we suppose that W is small relative to $E[L_M|X]$ in the decomposition of L_M, and introduce a new random variable L_M^ϵ with $\epsilon \in (0,1]$ by the following equation:

$$L_M^\epsilon = E[L_M|X] + \epsilon W = E[L_M|X] + \epsilon(L_M - E[L_M|X]).$$

Hence, $q_\alpha(L_M^\epsilon)$ can be approximated by a Taylor expansion around $E[L_M|X]$, i.e. $\epsilon = 0$. Particularly, the expansion up to the term of ϵ^2 with $\epsilon = 1$, we have the following approximate expression of $q_\alpha(L_M)$.

$$q_\alpha(L_M) \simeq q_\alpha(E[L_M|X]) + \left.\frac{dq_\alpha(L_M^\epsilon)}{d\epsilon}\right|_{\epsilon=0} + \frac{1}{2}\left.\frac{d^2 q_\alpha(L_M^\epsilon)}{d\epsilon^2}\right|_{\epsilon=0}. \tag{2.1}$$

Under Assumption 2.1, the first and third terms on the right-hand side of (2.1) are given by Theorem 2.1 and 2.2 below, respectively.

Theorem 2.1. *Let $q_\alpha(E[L_M|X])$ be the α-quantile of the loss rate distribution $E[L_M|X]$. Under Assumption 2.1-(ii), this α-quantile is given by*

$$q_\alpha(E[L_M|X]) = E[L_M|X = x_{1-\alpha}] = \frac{\sum_{i=1}^M E[U_i|X = x_{1-\alpha}]A_i}{\sum_{i=1}^M A_i},$$

where $x_{1-\alpha}$ is $1-\alpha$ quantile of X, namely $q_{1-\alpha}(X) = \inf\{y : P(X \le y) \ge 1-\alpha\}$.

Proof. See Proposition 4 in Gordy[8]. □

Hereafter, let us call $\Delta q_\alpha(L_M) = q_\alpha(L_M) - q_\alpha(E[L_M|X])$ the granularity adjustment of $q_\alpha(L_M)$. Then, we have the following theorem, which provides an approximation $\Delta\hat{q}_\alpha(L_M)$ for $\Delta q_\alpha(L_M)$, where only $\frac{1}{2}\left.\frac{d^2 q_\alpha(L_M^\epsilon)}{d\epsilon^2}\right|_{\epsilon=0}$ is evaluated on the right hand side of (2.1).

Theorem 2.2. *The granularity adjustment of $q_\alpha(L_M)$ is approximated by*

$$\Delta q_\alpha(L_M) \simeq \Delta\hat{q}_\alpha(L_M) = -\frac{1}{2f_X(x)}\frac{d}{dx}\left.\left(\frac{V[L_M|X=x]f_X(x)}{dE[L_M|X=x]/dx}\right)\right|_{x=x_{1-\alpha}}, \tag{2.2}$$

where $f_X(x)$ denotes the density function of X, $V[Y|X=x]$ does the conditional variance of a random variable Y given $X = x$, and the right-hand side of (2.2) is assumed to be finite.

Proof. See Section 2 in Ando[1]. □

We note Proposition 3 in Gordy[8] essentially shows that $q_\alpha(L_M) - q_\alpha(E[L_M|X]) \to 0$ as $M \to \infty$, namely the granularity adjustment term vanishes as the number of obligors increases to infinity under Assumption 2.1-(i) and the assumption ($\mathcal{A}$-2) in Gordy [8] (implying that when the number of obligors is large enough, the share of individual obligor's exposure becomes negligible). In practice, however, it is quite important to incorporate the granularity adjustment to consider concentration risks on large individual obligors.

2.1.3 One-factor Merton model

Following the framework of the Merton model, we assume that a default of obligor i occurs when a random variable R_i^* representing i's rate of asset return falls below a certain threshold in one year. Moreover, we put the next assumption on the obligor i's rate of asset return R_i^*.

Assumption 2.2. *Let R_i denote the standardized rate of asset return of obligor i defined by $R_i = \frac{R_i^* - E[R_i^*]}{\sqrt{V[R_i^*]}}$.*

Then, R_i is given by

$$R_i = \rho_i X + \sqrt{1-\rho_i^2}\xi_i, \qquad i = 1, \dots, M, \tag{2.3}$$

where X and ξ_i are mutually independent random variables following standard normal distributions and $E[\xi_i \xi_j] = 0,\ i \neq j$.

It is easily seen that R_i follows the standard normal distribution. Hereafter, this model is called one-factor Merton model, and ρ_i is referred to the factor loading of obligor i. Also, $\rho_i \rho_j$ expresses the correlation between R_i and R_j.

Next, let p_i be the default probability of obligor i within one year. Then, given p_i, the threshold which obligor i defaults if R_i is less than or equal to is expressed as $\Phi^{-1}(p_i)$, where Φ^{-1} denotes the inverse of the standard normal distribution function Φ.

Moreover, the loss rate U_i of an obligor i is expressed as $U_i = Q_i D_i$ using the loss rate at default Q_i and the default indicator function D_i for the obligor i, where $D_i = 1_{\{R_i \le \Phi^{-1}(p_i)\}}$ is 1 at default and 0 otherwise. We also note that the loss rate of an entire loan portfolio is given as

$$L_M = \sum_{i=1}^{M} w_i U_i = \sum_{i=1}^{M} w_i Q_i D_i, \tag{2.4}$$

where $w_i = A_i / \sum_{i=1}^{M} A_i$ is the weight of obligor i in the loan portfolio.

Also, we put the following condition.

Assumption 2.3. *For any obligor i, Q_i and D_i are independent conditioned on X.*

2.1.4 Closed-form expression of the α-quantile of $E[L_M|X]$

Using the default indicator function $D_i = 1_{\{R_i \le \Phi^{-1}(p_i)\}}$, the conditional default probability given $X = x$ denoted by $p_i(x)$ is obtained as follows:

$$p_i(x) = P\{R_i \le \Phi^{-1}(p_i) | X = x\} = \Phi\left(\frac{\Phi^{-1}(p_i) - \rho_i x}{\sqrt{1-\rho_i^2}} \right). \tag{2.5}$$

Therefore, we have the α quantiles of $E[L_M|X]$ as the following formula by applying Theorem 2.1, (2.4) and (2.5), with Assumption 2.3 (independence of Q_i and D_i conditioned on X): With a notation $\mu_i(x) := E[Q_i | X = x]$,

$$\begin{aligned} q_\alpha(E[L_M|X]) = E[L_M | X = x_{1-\alpha}] = E[L_M | X = \Phi^{-1}(1-\alpha)] &= \sum_{i=1}^{M} w_i E[Q_i D_i | X = \Phi^{-1}(1-\alpha)] \\ &= \sum_{i=1}^{M} w_i E[Q_i | X = \Phi^{-1}(1-\alpha)] E[D_i | X = \Phi^{-1}(1-\alpha)] \\ &= \sum_{i=1}^{M} w_i \mu_i(\Phi^{-1}(1-\alpha)) p_i\left(\Phi^{-1}(1-\alpha)\right) \\ &= \sum_{i=1}^{M} w_i \mu_i(\Phi^{-1}(1-\alpha)) \Phi\left(\frac{\Phi^{-1}(p_i) - \rho_i \Phi^{-1}(1-\alpha)}{\sqrt{1-\rho_i^2}} \right), \end{aligned} \tag{2.6}$$

2.1.5 Granularity adjustment

An approximation of the granularity adjustment term based on Theorem 2.2 is obtained as follows:

$$
\begin{aligned}
\Delta\hat{q}_\alpha(L_M) &= -\frac{1}{2\phi(x)}\frac{d}{dx}\left(\frac{V[L_M|X=x]\phi(x)}{l'(x)}\right)\Bigg|_{x=\Phi^{-1}(1-\alpha)} \\
&= -\frac{1}{2l'(x)}\left(v'(x) - v(x)\left(\frac{l''(x)}{l'(x)} + x\right)\right)\Bigg|_{x=\Phi^{-1}(1-\alpha)},
\end{aligned}
\tag{2.7}
$$

where $\phi(x)$ denotes the density function of the standard normal distribution, and

$$
\begin{aligned}
l(x) &= \sum_{i=1}^{M} w_i\mu_i(x)p_i(x), \quad l'(x) = \sum_{i=1}^{M} w_i[\mu_i'(x)p_i(x) + \mu_i(x)p_i'(x)], \\
l''(x) &= \sum_{i=1}^{M} w_i[\mu_i''(x)p_i(x) + 2\mu_i'(x)p_i'(x) + \mu_i(x)p_i''(x)], \\
v(x) &= V[L_M|X=x] = \sum_{i=1}^{M} w_i^2 p_i(x)[\mu_i^2(x)(1-p_i(x)) + \sigma_i^2(x)], \qquad (2.8) \\
v'(x) &= \sum_{i=1}^{M} w_i^2\{p_i'(x)[\mu_i^2(x)(1-2p_i(x)) + \sigma_i^2(x)] \\
&\quad + 2p_i(x)[\mu_i(x)\mu_i'(x)(1-p_i(x)) + \sigma_i(x)\sigma_i'(x)]\}, \\
p_i'(x) &= -\sqrt{\frac{\rho_i^2}{1-\rho_i^2}}\,\phi\left(\frac{\Phi^{-1}(p_i) - \rho_i x}{\sqrt{1-\rho_i^2}}\right), \\
p_i''(x) &= -\frac{\rho_i^2}{1-\rho_i^2}\frac{\Phi^{-1}(p_i) - \rho_i x}{\sqrt{1-\rho_i^2}}\,\phi\left(\frac{\Phi^{-1}(p_i) - \rho_i x}{\sqrt{1-\rho_i^2}}\right).
\end{aligned}
$$

Here, $l(x)$ with $x = \Phi^{-1}(1-\alpha)$ represents $q_\alpha(E[L_M|X])$ in (2.6), and $w_i = A_i/\sum_{i=1}^{M} A_i$; $p_i(x)$ is given by (2.5); $\sigma_i^2(x) := V(Q_i|X=x)$; p_i is the default probability of obligor i in one year; ρ_i is the factor loading of obligor i in (2.3).

In addition, we obtain (2.8) by the following calculation:

$$
\begin{aligned}
v(x) = V[L_M|X=x] &= V\left(\sum_{i=1}^{M} w_i U_i | X=x\right) = \sum_{i=1}^{M} w_i^2 V(Q_i D_i|X=x) \\
&= \sum_{i=1}^{M} w_i^2\left[E(Q_i^2 D_i^2|X=x) - E(Q_i D_i|X=x)^2\right] \\
&= \sum_{i=1}^{M} w_i^2\left[E(Q_i^2|X=x)E(D_i^2|X=x) - (\mu_i(x)p_i(x))^2\right] \\
&= \sum_{i=1}^{M} w_i^2\left[(\mu_i^2(x) + \sigma_i^2(x))p_i(x) - (\mu_i(x)p_i(x))^2\right] \\
&= \sum_{i=1}^{M} w_i^2 p_i(x)[\mu_i^2(x)(1-p_i(x)) + \sigma_i^2(x)],
\end{aligned}
$$

where we use Assumption 2.1-(i) (independence of U_i $(i, \ldots, M)$ conditioned on X), as well as Assumption 2.3 (independence of Q_i and D_i conditioned on X).

2.2 Regulatory and Economic capital credit risks

Using the approximate quantile of the loss rate distribution obtained in the previous section, this subsection formulates the credit risk amount corresponding to the regulatory and economic capital defined in Section 1.

First, let us recall that the regulatory capital is defined as "the minimum amount of funds required by financial authorities to ensure soundness of the entire financial system." More concretely, following Basel III [3], we regard the credit risk amount, $risk^{RC}$ in (2.14) below as the regulatory capital in our framework. Hereafter, we call it *regulatory capital credit risk*. We also remark that the regulatory capital should be greater than or equal to 8% of risk-weighted assets (RWAs), where RWAs are calculated by the Standardized Approach (SA) or Internal Rating Based Approach (IRB). (See [3] for the details of the SA and IRB approach.)

Next, we recall that economic capital is defined as "the funds that financial institutions should prepare for potential losses arising from the risks associated with their businesses." Following Basel III [2],we regard the credit risk amount $risk^{EC}$ in (2.15) as the economic capital in our framework, which is equivalent to the difference of the right hand side of (2.1) from the expected loss of the loan portfolio (EL): Namely, an approximation for $q_\alpha(L_M)$, i.e., the α-quantile of the loss rate distribution L_M for the entire loan portfolio minus its expected loss EL in (2.9). Hereafter, we call $risk^{EC}$ *economic capital credit risk*.

If the credit risk amount $q_\alpha(L_M)$ is all we want to get, it is obtained by a simulation method. However, since we need to repeatedly calculate the credit risk amount to obtain optimal portfolios numerically in the following sections, we use a closed-form approximation to reduce the computational burden.

2.2.1 Expected Loss (EL)

Firstly, we introduce the expected loss (EL), the average expected loss in one year. In our framework, the expected loss of a loan portfolio with M obligors is given as follows: Since $EL_i = E[A_i U_i] = A_i E[Q_i D_i] = A_i(\mu_i p_i + \sigma^i_{Q,D})$ with a notation $\sigma^i_{Q,D} := Cov(Q_i, D_i)$, that is the covariance between Q_i and D_i,

$$EL = \sum_{i=1}^{M} EL_i = \sum_{i=1}^{M} A_i(\mu_i p_i + \sigma^i_{Q,D}), \tag{2.9}$$

where A_i is the loan exposure to obligor i, μ_i is the mean of Q_i, and p_i is the default probability of obligor i in one year.

2.2.2 Unexpected Loss (UL)

Next, let us describe the unexpected loss (UL), that is the maximum loss deviated from the expected loss (EL) within one year. In our framework, we obtain it by subtracting EL from the maximum loss estimated under a certain probability. More concretely, with the α quantile of $E[L_M|X]$, i.e., $q_\alpha(E[L_M|X])$ given by (2.6), which is the first term in the right hand side of

(2.1), the unexpected loss of a loan portfolio with M obligors is given in the following:

$$\begin{aligned} UL &= \sum_{i=1}^{M} UL_i = A\ q_\alpha(E[L_M|X]) - \sum_{i=1}^{M} EL_i \\ &= \sum_{i=1}^{M} \left(Aw_i\mu_i(\Phi^{-1}(1-\alpha))\Phi\left(\frac{\Phi^{-1}(p_i)-\rho_i\Phi^{-1}(1-\alpha)}{\sqrt{1-\rho_i^2}}\right) - EL_i \right) \\ &= \sum_{i=1}^{M} A_i \left(\mu_i(\Phi^{-1}(1-\alpha))\Phi\left(\frac{\Phi^{-1}(p_i)-\rho_i\Phi^{-1}(1-\alpha)}{\sqrt{1-\rho_i^2}}\right) - [\mu_i p_i + \sigma^i_{Q,D}] \right), \end{aligned}$$

where

$$\begin{aligned} UL_i &= A_i\mu_i(\Phi^{-1}(1-\alpha))\Phi\left(\frac{\Phi^{-1}(p_i)-\rho_i\Phi^{-1}(1-\alpha)}{\sqrt{1-\rho_i^2}}\right) - EL_i \\ &= A_i \left(\mu_i(\Phi^{-1}(1-\alpha))\Phi\left(\frac{\Phi^{-1}(p_i)-\rho_i\Phi^{-1}(1-\alpha)}{\sqrt{1-\rho_i^2}}\right) - [\mu_i p_i + \sigma^i_{Q,D}] \right), \end{aligned} \tag{2.10}$$

$A = \sum_{i=1}^{M} A_i$ is the total portfolio exposure, $w_i = A_i / \sum_{i=1}^{M} A_i$, and ρ_i is the factor loading of obligor i in (2.3).

2.2.3 Regulatory capital credit risk ($risk^{RC\_IRB}$; $risk^{RC\_SA}$; $risk^{RC}$)

First, let us formulate the IRB-based credit risk amount $risk^{RC\_IRB}$ following Basel III [3]. Particularly, Basel III empirically reflects that long-term loans have higher credit risk than short-term loans and that high-rated obligors have more room for downgrading than low-rated obligors. This is called maturity adjustment m_adj_i, which is an increasing function of an effective maturity m_i and a decreasing function of one-year default probability p_i for $m_i \geq 2.5$. (See [3] for details of the maturity adjustment.) Then, the unexpected loss after maturity adjustment of a loan portfolio with M obligors denoted by $risk^{RC\_IRB}$ is given as follows:

$$\begin{aligned} risk^{RC\_IRB} &= \sum_{i=1}^{M} risk_i^{RC\_IRB} = \sum_{i=1}^{M} UL_i \times m\_adj_i \\ &= \sum_{i=1}^{M} A_i \left(\mu_i(\Phi^{-1}(1-\alpha))\Phi\left(\frac{\Phi^{-1}(p_i)-\rho_i\Phi^{-1}(1-\alpha)}{\sqrt{1-\rho_i^2}}\right) - [\mu_i p_i + \sigma^i_{Q,D}] \right) \times m\_adj_i, \end{aligned} \tag{2.11}$$

$$m\_adj_i = \frac{1+(m_i-2.5)b_i}{1-1.5b_i}, \qquad b_i = \{0.11852 - 0.05478 \times log(p_i)\}^2, \tag{2.12}$$

where $risk_i^{RC\_IRB} = UL_i \times m\_adj_i$ with (2.10), and the constants $-2.5, -1.5, 0.11852, -0.05478$ in m_adj_i and b_i are given in p.116 of [3]. In our numerical analysis, we will set $\alpha = 0.999$ and use the formula (3.1) in Section 3.2 to determine ρ_i based on the Basel III framework.

Next, we formulate the SA-based credit risk amount $risk^{RC\_SA}$. For calculation of $risk^{RC\_SA}$, we need realistic external ratings consistent with internal ratings for all obligors i, which is difficult to obtain. Hence, we adopt an alternative simple method, that is multiplying $risk^{RC\_IRB}$ by some constant. Concretely, when using a constant β common among all obligors, we obtain $risk^{RC\_SA}$ as follows:

$$risk^{RC\_SA} = \beta \times risk^{RC\_IRB} = \beta \sum_{i=1}^{M} risk_i^{RC\_IRB}. \tag{2.13}$$

With this $risk^{RC\_SA}$, we formulate the regulatory capital credit risk, $risk^{RC}$. Moreover, in calculation of $risk^{RC}$, we also need to take the latest regulatory requirement into account, which requires that $risk^{RC}$ cannot fall below a certain ratio of $risk^{RC\_SA}$. Hence, using a constant κ as this ratio, we express $risk^{RC}$ as follows:

$$\begin{aligned} risk^{RC} &= \max\{risk^{RC\_IRB}, \kappa \times risk^{RC\_SA}\} \\ &= \max\{risk^{RC\_IRB}, \kappa \times \beta \times risk^{RC\_IRB}\} \\ &= \max\{1, \kappa \times \beta\} \times risk^{RC\_IRB}. \end{aligned} \tag{2.14}$$

However, we will apply different β to different business units for our numerical analysis in Section 3. We will formulate the case in Section 2.2.5, and explain how to set those β as well as the constant κ in detail at the end of Section 3.3.

2.2.4 Economic capital credit risk ($risk^{EC}$)

Following Basel III [2],this subsection formulates the economic capital credit risk, $risk^{EC}$, that is the capital which a bank needs to protect against unexpected future losses. More concretely, it is obtained as the difference of the α quantile of the loss rate distribution of the loan portfolio from the expected loss (EL), i.e., $q_\alpha(L_M)$ on the left-hand side of (2.1) minus EL. Further, to reduce the computational burden of portfolio optimization, we use an analytical approximation given on the right-hand side of (2.1) with maturity adjustment m_adj_i. Precisely, we obtain it by adding (2.7), an approximation of the granularity adjustment term $\Delta q_\alpha(L_M)$ to the $risk^{RC\_IRB}$ (2.11) as follows: By $risk_i^{RC\_IRB} = UL_i \times m\_adj_i$ with (2.10), and the same definitions of notations as in Section 2.1.5,

$$\begin{aligned} risk^{EC} &= risk^{RC\_IRB} - \frac{A}{2l'(x)}\left(v'(x) - v(x)\left(\frac{l''(x)}{l'(x)} + x\right)\right)\Bigg|_{x=\Phi^{-1}(1-\alpha)} \\ &= \sum_{i=1}^{M} risk_i^{RC\_IRB} - \frac{A}{2l'(x)}\left(v'(x) - v(x)\left(\frac{l''(x)}{l'(x)} + x\right)\right)\Bigg|_{x=\Phi^{-1}(1-\alpha)}. \end{aligned} \tag{2.15}$$

We remark that it is also acceptable for banks to calculate the unexpected loss (UL) of their loan portfolio based on the methods such as Monte Carlo simulations, which are different from (2.10) in $risk^{RC\_IRB}$ above.

2.2.5 Credit risk by segment (industrial sector and business unit(BU))

Let $\gamma(j,k)$ be a set of obligors belonging to a segment (industrial sector j within business unit k) of a loan portfolio. As will be explained in the following subsection 3.1, each business unit (BU) prepares its business strategy separately. Accordingly, we use different constants, β_k ($k = 1, 2, \cdots, n$) for different business units, BU (k) ($k = 1, 2, \cdots, n$), while β_k is the same among all obligors within each BU. Then, the expected loss by segment $EL_{j,k}$, the credit risk by segment $risk_{j,k}^{RC}$ and $risk_{j,k}^{EC}$ are given below:

$$EL_{j,k} = \sum_{i=1, i\in\gamma(j,k)}^{M} EL_i = \sum_{i=1, i\in\gamma(j,k)}^{M} A_i(\mu_i p_i + \sigma_{Q,D}^i), \tag{2.16}$$

$$risk_{j,k}^{RC} = \max\{1, \kappa \times \beta_k\} \times \sum_{i=1, i\in\gamma(j,k)}^{M} risk_i^{RC\_IRB}, \tag{2.17}$$

$$risk_{j,k}^{EC} = \sum_{i=1, i\in\gamma(j,k)}^{M} risk_i^{RC\_IRB} + \Delta\hat{q}_\alpha(L_M)_{j,k}, \tag{2.18}$$

where $risk_i^{RC\_IRB} = UL_i \times m\_adj_i$ with (2.10) and (2.12), and

$$\Delta\hat{q}_\alpha(L_M)_{j,k} = -\frac{A_{j,k}}{2\tilde{l}'(x)}\left(\tilde{v}'(x) - \tilde{v}(x)\left(\frac{\tilde{l}''(x)}{\tilde{l}'(x)} + x\right)\right)\Bigg|_{x=\Phi^{-1}(1-\alpha)}, \tag{2.19}$$

$$A_{j,k} = \sum\nolimits_{i=1, i\in\gamma(j,k)}^{M} A_i, \tag{2.20}$$

$$\tilde{l}(x) = \sum\nolimits_{i=1, i\in\gamma(j,k)}^{M} w_i \mu_i(x) p_i(x),$$

$$\tilde{l}'(x) = \sum\nolimits_{i=1, i\in\gamma(j,k)}^{M} w_i[\mu_i'(x) p_i(x) + \mu_i(x) p_i'(x)],$$

$$\tilde{l}''(x) = \sum\nolimits_{i=1, i\in\gamma(j,k)}^{M} w_i[\mu_i''(x) p_i(x) + 2\mu_i'(x) p_i'(x) + \mu_i(x) p_i''(x)],$$

$$\tilde{v}(x) = \sum\nolimits_{i=1, i\in\gamma(j,k)}^{M} w_i^2 p_i(x)[\mu_i^2(x)(1 - p_i(x)) + \sigma_i^2(x)], \tag{2.21}$$

$$\tilde{v}'(x) = \sum\nolimits_{i=1, i\in\gamma(j,k)}^{M} w_i^2\{p_i'(x)[\mu_i^2(x)(1 - 2p_i(x)) + \sigma_i^2(x)] + 2p_i(x)[\mu_i(x)\mu_i'(x)(1 - p_i(x)) + \sigma_i(x)\sigma_i'(x)]\}.$$

Here, the definitions of notations are the same as in Section 2.1.5. Namely, $w_i = A_i / \sum_{i=1}^{M} A_i$, and $p_i(x)$ is given by (2.5). Also, the expression in (2.21) is similarly obtained as in (2.8).

2.3 Portfolio optimization

Based on the regulatory capital credit risk $risk_{j,k}^{RC}$ obtained as (2.17) and economic capital credit risk $risk_{j,k}^{EC}$ in (2.18), we seek a portfolio that maximizes profits by changing exposure to segments $A_{j,k}$ in (2.20) of the previous subsection. See Problem 2.1 below for the details.

2.3.1 Exposure change

Let $\tilde{A}_i = A_i + \Delta A_i$ and $\tilde{w}_i = \tilde{A}_i / \sum_{i=1}^{M} \tilde{A}_i$. Then, the exposure to segment $\gamma(j,k)$ is given as $\tilde{A}_{j,k} = \sum_{i=1, i\in\gamma(j,k)}^{M} \tilde{A}_i$. Furthermore, the exposures aggregated by industrial sector $j(\leq m)$, BU $k(\leq n)$ and overall are given as $\tilde{A}_j$, $\tilde{A}_k$ and $\tilde{A}$, respectively:

$$\tilde{A}_j = \sum_{k=1}^{n} \tilde{A}_{j,k}, \quad \tilde{A}_k = \sum_{j=1}^{m} \tilde{A}_{j,k}, \quad \tilde{A} = \sum_{j=1}^{m}\sum_{k=1}^{n} \tilde{A}_{j,k}.$$

2.3.2 Expected loss after exposure change

We consider a expected loss when an exposure of obligor i changes. Replacing A_i in (2.16) with $\tilde{A}_i$, we obtain $\tilde{EL}_{j,k}$. Furthermore, the expected loss aggregated by industrial sector $j(\leq m)$, BU $k(\leq n)$ and overall are given as $\tilde{EL}_j$, $\tilde{EL}_k$ and $\tilde{EL}$, respectively:

$$\tilde{EL}_j = \sum_{k=1}^{n} \tilde{EL}_{j,k}, \quad \tilde{EL}_k = \sum_{j=1}^{m} \tilde{EL}_{j,k}, \quad \tilde{EL} = \sum_{j=1}^{m}\sum_{k=1}^{n} \tilde{EL}_{j,k}.$$

2.3.3 Credit risk after exposure change

We consider a credit risk amount when an exposure of obligor i changes. Replacing (A_i, w_i) in (2.17) and (2.18) with $(\tilde{A}_i, \tilde{w}_i)$, we obtain $\tilde{risk}_{j,k}^{RC}$ and $\tilde{risk}_{j,k}^{EC}$. Furthermore, the regulatory and economic capital credit risks aggregated by industrial sector $j(\leq m)$, BU $k(\leq n)$ and overall are given as (2.22) and (2.23), respectively:

$$\tilde{risk}_j^{RC} = \sum_{k=1}^{n} \tilde{risk}_{j,k}^{RC}, \quad \tilde{risk}_k^{RC} = \sum_{j=1}^{m} \tilde{risk}_{j,k}^{RC}, \quad \tilde{risk}^{RC} = \sum_{j=1}^{m}\sum_{k=1}^{n} \tilde{risk}_{j,k}^{RC}, \tag{2.22}$$

$$\tilde{risk}_j^{EC} = \sum_{k=1}^{n} \tilde{risk}_{j,k}^{EC}, \quad \tilde{risk}_k^{EC} = \sum_{j=1}^{m} \tilde{risk}_{j,k}^{EC}, \quad \tilde{risk}^{EC} = \sum_{j=1}^{m}\sum_{k=1}^{n} \tilde{risk}_{j,k}^{EC}. \tag{2.23}$$

2.3.4 Profit after exposure change

We consider a profit when an exposure of obligor i changes. Let p_rate_i be a profit rate from obligor i, which consists of base rate (b_rate_i), margin spread (m_sp_i), funding rate (f_rate_i), and expected loss rate (e_loss_i). Multiplying the exposure $\tilde{A}_i$ and the p_rate_i provides $\tilde{profit}_i$. Then, the profit by segment $\tilde{profit}_{j,k}$ is obtained by aggregating $\tilde{profit}_i$ within $\gamma(j,k)$.

$$p\_rate_i = b\_rate_i + m\_sp_i - f\_sp_i - e\_loss_i, \tag{2.24}$$
$$\tilde{profit}_i = \tilde{A}_i \times p\_rate_i, \qquad \tilde{profit}_{j,k} = \sum\nolimits_{i=1, i\in\gamma(j,k)}^{M} \tilde{profit}_i.$$

Furthermore, the profits aggregated by industrial sector $j(\leq m)$, BU $k(\leq n)$ and overall are given in (2.25):

$$\tilde{profit}_j = \sum_{k=1}^{n} \tilde{profit}_{j,k}, \quad \tilde{profit}_k = \sum_{j=1}^{m} \tilde{profit}_{j,k}, \quad \tilde{profit} = \sum_{j=1}^{m}\sum_{k=1}^{n} \tilde{profit}_{j,k}. \tag{2.25}$$

We remark that since p_rate_i above does not include all the costs such as labor costs, the segment profit $\tilde{profit}_{j,k}$ is an internal measure, which is not equivalent to an accounting profit.

2.3.5 Portfolio optimization with constraints

The constraints for financial institutions are closely related to the risk appetite framework (RAF) introduced in [7] published by the Financial Stability Board (FSB) in 2013. In order to monitor and control risk appropriately, it defines risk capacity, risk appetite and risk limits as key components in the following:

The risk capacity is defined as 'the maximum level of risks the financial institution can assume given its current level of resources.' *The risk appetite* is defined as 'the aggregate level and types of risk a financial institution is willing to assume within its risk capacity to achieve its strategic objectives and business plan.' *The risk limits* are defined as 'quantitative measures based on forward looking assumptions that allocate the financial institution's aggregate risk appetite statement (e.g. measure of loss or negative events) to ..., specific risk categories, concentrations, ...'

Most of international banks use the RAF to refine their business plan and set constraints accordingly. Hence, reflecting the RAF and a constraint on the lending market, we prepare the following four constraints:

1. Total risk constraint as the regulatory required capital, which corresponds to *the risk capacity* in RAF and keeps the total credit risk amount within the regulatory capital.
2. Risk constraint on each business unit (BU), k as the internal requirement, which corresponds to *the risk appetite* in RAF, particularly for each BU k, and keeps the BU k's credit risk amount within the BU k's risk appetite.
3. Risk constraint on each segment (j,k) as the internal requirement, which corresponds to *the risk limits* in RAF and keeps each segment (j,k)'s credit risk amount within the risk limit.

4. Loan market constraint, which keeps the exposure changes in each segment (j,k) within a certain percentage of the outstanding exposure.

In addition, we prepare different constraint cases according to which credit risk (i.e., regulatory or economic capital credit risk) is used for the constraint conditions.

Under these constraints, we compute a portfolio that maximizes profits by changing the exposure of the top six risk segments with respect to the regulatory capital credit risk, $risk^{RC}$ for each domestic and foreign BU. Concretely, we have the following optimization problem to solve.

Problem 2.1. *Let m be the number of industrial sectors and n be the number of business units(BUs). In the equation (2.26) below that shows our optimization problem, we prepare constants on $\tilde{risk}$, $\tilde{risk}_k$, $\tilde{risk}_{j,k}$ and $(A_{j,k}+\Delta A_{j,k})$. Moreover, while we use regulatory capital credit risks for $\tilde{risk}$, that is denoted by $\tilde{risk}^{RC}$, we apply both regulatory and economic capital credit risks to $\tilde{risk}_k$ and $\tilde{risk}_{j,k}$, which are denoted by $\tilde{risk}_k^{RC}$, $\tilde{risk}_{j,k}^{RC}$ and $\tilde{risk}_k^{EC}$, $\tilde{risk}_{j,k}^{EC}$, respectively.*

Here, $\tilde{risk}_{j,k}^{RC}$ and $\tilde{risk}_{j,k}^{EC}$ are obtained by replacing (A_i, w_i) in (2.17) and (2.18) with $(\tilde{A}_i, \tilde{w}_i)$, and $\tilde{risk}_k^{RC}$ and $\tilde{risk}_k^{EC}$ are given accordingly.

Those different constraint cases we use are named as RC(regulatory capital) for all, EC(economic capital) & RC(regulatory capital) for IR(internal requirement), and EC(economic capital) for IR(internal requirement), which are summarized in Table 1 below.

Hence, the portfolio optimization problem with constraints is given as

$$
\begin{aligned}
&\textit{maximize} && \sum_{j=1}^{m}\sum_{k=1}^{n}\tilde{profit}_{j,k}, \qquad w.r.t.\quad \Delta A_{j,k},\\
&\textit{subject to} && \tilde{risk} \le RiskCapasity,\\
& && \tilde{risk}_k \le RiskAppetite_k, \qquad k=1,\dots,n,\\
& && \tilde{risk}_{j,k} \le RiskLimit, \qquad j=1,\dots,m, \quad k=1,\dots,n,\\
& && (1-\delta)A_{j,k} \le (A_{j,k}+\Delta A_{j,k}) \le (1+\delta)A_{j,k}, \qquad 0\le\delta\le 1,
\end{aligned} \tag{2.26}
$$

where $RiskCapasity$ stands for the regulatory required capital, $RiskAppetite_k$ for the BU k's risk appetite, and $RiskLimit$ for the upper risk limit applied to the segment (j,k);we set $RiskLimit$ to be the same for all (j,k). Also, the exposure change ratio δ denotes a loan market constraint. We will explain $RiskCapacity$, $RiskAppetite_k$, $RiskLimit$ and δ to the detail in later Section 3.3.

Table1 Constraint cases

Constraints case name	Regulatory requirement	Internal requirement (IR)	
	Total risk constraint	BU risk constraint	Segment risk constraint
I. RC for all	$\tilde{risk}^{RC} \le RiskCapacity$	$\tilde{risk}_k^{RC} \le RiskAppetite_k$	$\tilde{risk}_{j,k}^{RC} \le RiskLimit$
II. EC&RC for IR	$\tilde{risk}^{RC} \le RiskCapacity$	$\tilde{risk}_k^{RC} \le RiskAppetite_k$	$\tilde{risk}_{j,k}^{EC} \le RiskLimit$
III. EC for IR	$\tilde{risk}^{RC} \le RiskCapacity$	$\tilde{risk}_k^{EC} \le RiskAppetite_k$	$\tilde{risk}_{j,k}^{EC} \le RiskLimit$

We remark that since the business strategy in a typical international bank is developed by each business unit(BU), the cross-business-unit constraint, i.e., the constraint on $\tilde{risk}_j$, the sum of $\tilde{risk}_{j,k}$ over all k for each j is not considered in Problem 2.1.

Next, let us briefly explain Table 1. Firstly, for all cases, I, II and III, the regulatory capital credit risk, $\tilde{risk}^{RC}$ should be used for the total risk constraint by the regulatory requirement. Then, in case I, we apply $\tilde{risk}^{RC}$ for all the internal requirement (IR) as well. On the contrary, in case II, we use the economic capital credit risk, $\tilde{risk}^{EC}_{j,k}$ for the segment risk constraint to take concentration risk into account, since it tends to occur frequently in a segment. In case III, we also apply $\tilde{risk}^{EC}_{k}$, i.e., the sum of $\tilde{risk}^{EC}_{j,k}$ over all j for each k to the BU risk constraint.

As it seems not meaningful to consider concentration risk on each BU without doing it on each segment, which is likely to have more concentration risk than the business unit it belongs to, we do not consider the case of $\tilde{risk}^{EC}_{k}$ for the BU risk constraint and $\tilde{risk}^{RC}_{j,k}$ for the segment risk constraint.

Moreover, as will be explained to the detail in Section 3.3, we set $RiskAppetite_k$ to be lower than the sum of $RiskLimit$ over all k, i.e., $m \times RiskLimit$.

3 Numerical analysis

In this section, first we explain the characteristics of a typical loan portfolio of international banks and construct a loan portfolio with those characteristics, which is used for our numerical analysis. Then, we explain the four constraints for optimization in the analysis. Finally, we estimate and investigate the optimal portfolio.

3.1 Loan portfolio of international banks

Let us explain the characteristics of a typical loan portfolio of international banks. International banks often have multiple business units(BUs), due to the difference of business environments among regions. Also, the loan portfolio of international banks has exposure concentration on industrial sectors within each BU and obligors in each industrial sector. The details are explained in the following.

- Preparing business strategy for each BU : The international banks often have domestic and foreign BUs because the economic environment and targeted customers are different between domestic and foreign regions. Hence, each BU prepares its business strategy separately, and each strategy is further divided into sectors and products. Accordingly, the details of our loan portfolio will be shown in Table 2 below.
- Industrial sector concentration within BU : Although the composition of a bank's loan portfolio depends on the industrial structure of the country to which the bank belongs, the portfolio tends to be concentrated on specific industrial sectors within BU.
  As a typical characteristic of international banks, the loan exposure concentrates on 'Industrials' that consists of construction, machinery manufacturing, general trading companies, and transportation. The loan exposure also concentrates on 'Consumer Discretionary' that consists of automobiles, home appliances, and apparel.
  Moreover, since international banks participate in the development of infrastructure around the world, they have a lot of exposure to 'Utilities.' They also have many loans to domestic 'Real Estate' businesses. The loan terms of 'Real Estate' and 'Utilities' tend to be relatively long. These tendency will be also seen in Table 2 below.
- Obligor concentration : Since the bank supports the business expansion of customers through their strong relationship, the loan may be concentrated on an individual obligor in the long run. We can often see obligor concentration, as in an automobile company with many supporting industries such as parts manufacturers, a conglomerate centered on the real estate industry, and an energy company involved in a wide range of activities from extraction to the refining of oil and gas. The items b in Table 4 below provides an

example of concentration on one large individual obligor. Namely, 25% of the exposure is concentrated on one obligor, and the rest is equally allocated.

3.2 Settings of the loan portfolio

In this subsection, first we prepare the basic component of the loan portfolio with the characteristics of the loan portfolio of an international bank explained in the previous section. Particularly, we assume that there are two BUs ($n = 2$), where one is domestic ($k = 1$) and the other is foreign ($k = 2$). We also suppose that there are 12 industrial sectors ($m = 12$) listed in AppendixA, according to S&P's Global Industry Classification Standard (GICS). Thus, the loan portfolio is classified into 24 segments. The exposure of each segment $A_{j,k}$ is set by referring to the Bank of Japan(BOJ)'s loan statistics "Loans and Bills Discounted by Sector."

The regulatory capital credit risk $risk^{RC}_{j,k}$ is calculated by the formula (2.17) with parameters given in Table 3 below. The maturity of each segment $m_{j,k}$ is set as three years except for Real Estate and Utilities whose maturities are five years. The following Table 2 shows the exposure $A_{j,k}$, the regulatory capital credit risk $risk^{RC}_{j,k}$ and maturity $m_{j,k}$ of the top six segments with respect to $risk^{RC}_{j,k}$ for each BU. Also, the columns corresponding to "Subtotal/Weighted average" and "Total/Weighted average" show those for all 12 segments for each BU and for all 24 segments, respectively.

Table2 The loan portfolio by segment $\gamma(j, k)$: (exposure $A_{j,k}$: 100 million yen, regulatory capital credit risk $risk^{RC}_{j,k}$: 100 million yen, maturity $m_{j,k}$: year)

BU (k)	Industrial sector (j)	$A_{j,k}$	$risk^{RC}_{j,k}$	$m_{j,k}$
Domestic (1)	Industrials (1)	12,000	625	3.0
Domestic (1)	Consumer Discretionary (2)	8,000	487	3.0
Domestic (1)	Real Estate (3)	9,000	324	5.0
Domestic (1)	Materials (4)	6,000	307	3.0
Domestic (1)	Financials (5)	7,000	227	3.0
Domestic (1)	Utilities (6)	5,000	214	5.0
Domestic (1)	Subtotal/Weighted average	60,000	2,823	3.4
Foreign (2)	Industrials (1)	8,000	487	3.0
Foreign (2)	Utilities (6)	8,000	440	5.0
Foreign (2)	Consumer Discretionary (2)	4,000	282	3.0
Foreign (2)	Energy (10)	4,000	227	3.0
Foreign (2)	Information Technology (8)	3,000	206	3.0
Foreign (2)	Financials (5)	4,000	169	3.0
Foreign (2)	Subtotal/Weighted average	40,000	2,312	3.5
	Total/Weighted average	100,000	5,135	3.5

Next, we prepare the risk and profit parameters of the loan portfolio as follows: Firstly, due to the limitation of available data, we put the assumptions that $\mu_i(x) = \mu_i$, $\sigma_i^2(x) = \sigma_i^2$ and $\sigma^i_{Q,D} = 0$. (That is, $E(Q_i|X = x) = E(Q_i)$, $V(Q_i|X = x) = V(Q_i)$, $Cov(Q_i, D_i) = 0$, respectively.)

Moreover, we use the same risk and profit parameters for all $i \in \gamma(j, k)$ (i.e., for all obligors in the same segment), and follow the notation $\mu_{j,k}$ to denote μ_i for all i belonging to the same segment $\gamma(j, k)$, for instance. These data are as of March 2021 except for $\mu_{j,k}$ and $\sigma_{j,k}$. As for $\mu_{j,k}$ with $k = 1$ (domestic segment) and $\mu_{j,k}$, $\sigma_{j,k}$ with $k = 2$ (foreign segment), we use the most recent available data, namely those as of March 2020. Lastly, we take the datum for $\sigma_{j,k}$ with $k = 1$ (domestic segment) from Section 3.4 in Ando[1], which however, does not specify its date.

- $p_{j,k}$: The one-year default probability $p_{j,k}$ for all $i \in \gamma(j, k)$ is set based on "Table 19:

Global Corporate Default Rates By Industry, 1981-2021" in [21]. In Table 19, since only the average default probability within foreign segment $p_{j,2}$ is available, we set the average default probability within domestic segment $p_{j,1}$ as follows:

$$p_{j,1} = p_{j,2} \times \frac{domestic\_p}{foreign\_p},$$

where $domestic\_p$ and $foreign\_p$ are set as 0.04% and 0.08%, respectively, which are given as default probability of Japanese investment grade obligors in "Exhibit 2: Average Accumulate Default Probability, 1990-2021" of [16].

- $\mu_{j,k}$: The mean of loss rate at default for all $i \in \gamma(j,k)$, namely $\mu_{j,k}$ is set based on "Table 2: Average Recovery by non-financial Sector, 1987-2020" in [20].
  Particularly, due to the limitation of data, we set $\mu_{j,k} = 1 - recovery\ rate_{j,k}$, where $recovery\ rate_{j,k}$, the recovery rate for the segment $\gamma(j,k)$ is given in this Table 2, and use the same $\mu_{j,k}$ for both domestic and foreign BUs.
  Moreover, we remark that the recovery rate $\mu_{j,k}$ for the segment $\gamma(j,k)$ takes the collateral into account.
- $\sigma_{j,k}$: The standard deviation of loss rate at default for all $i \in \gamma(j,k)$, $\sigma_{j,k}$ is set based on "Table 1: Recovery Rates By Instrument Type, 1987-2020" in [20] except for domestic segment ($k = 1$). The $\sigma_{j,k}$ for domestic segment is set 25%, which is the same as in Section 3.4 of Ando[1].
- $\rho_{j,k}$: The factor loading for all $i \in \gamma(j,k)$, $\rho_{j,k}$ is calculated by the correlation formula for the risk weight function given in [3] as follows:

$$\begin{aligned}\rho_{j,k} &= 0.12 \times \frac{(1-\exp(-50 \times p_{j,k}))}{(1-\exp(-50))} + 0.24 \times \left(1 - \frac{(1-\exp(-50 \times p_{j,k}))}{(1-\exp(-50))}\right) \\ &= 0.24 - 0.12 \times \frac{(1-\exp(-50 \times p_{j,k}))}{(1-\exp(-50))}\end{aligned} \tag{3.1}$$

  where the factor loading $\rho_{j,k}$, a decreasing function of $p_{j,k}$ represents the degree of dependence on systematic risk factor X. That is, in case of high $p_{j,k}$, the dependence on the systematic risk factor X is low, and vice versa.
- $b\_rate_{j,k}$: The base rate for all $i \in \gamma(j,k)$, $b\_rate_{j,k}$ is equivalent to the credit risk of the segment, where we assume that $base\_rate_{j,k}$ is the same as default probability $p_{j,k}$.
- $m\_sp_{j,k}$: The margin spread for all $i \in \gamma(j,k)$, $m\_sp_{j,k}$ is set as 51 bp for domestic (k=1) and 102 bp for foreign (k=2) by referring to [15],[19],[14]. Concretely, since international banks usually revise their business plans on annual basis, we calculate the average margin spread for each BU k by using the spreads during the most recent two semi-annual periods reported in [15],[19],[14].
- $f\_rate_{j,k}$: The funding rate for all $i \in \gamma(j,k)$, $f\_rate_{j,k}$ is set as 0 bp for domestic (k=1) and 18 bp for foreign (k=2) by referring to [15],[19],[14]. Following the same procedure as in the margin spread, we calculate average funding rate for each BU k. (That is, we use the rates during the most recent two semi-annual periods reported in [15],[19],[14].)
- $e\_loss_{j,k}$: The expected loss rate for all $i \in \gamma(j,k)$, $e\_loss_{j,k}$ is calculated as the expected loss of the segment $\gamma(j,k)$ divided by the exposure of the segment$\gamma(j,k)$ as follows:

$$e\_loss_{j,k} = \mu_{j,k} p_{j,k}.$$

- $p\_rate_{j,k}$: The profit rate for all $i \in \gamma(j,k)$, $p\_rate_{j,k}$ is calculated by using the equation (2.24) as follows:

$$p\_rate_{j,k} = b\_rate_{j,k} + m\_sp_{j,k} - f\_rate_{j,k} - e\_loss_{j,k}.$$

- $risk_{j,k}^{RC}$: The regulatory capital credit risk for the segment $\gamma(j,k)$, $risk_{j,k}^{RC}$ is calculated by the formula (2.17) with the exposure $A_{j,k}$ and parameters $p_{j,k}, \mu_{j,k}, \sigma_{j,k}, \rho_{j,k}, \kappa, \beta_k$, where κ and β_k will be explained at the end of Section 3.3.
- $risk_{j,k}^{EC}$: The economic capital credit risk amount for the segment $\gamma(j,k)$, $risk_{j,k}^{EC}$ is calculated by the formula (2.18) with the exposure $A_{j,k}$ and parameters $p_{j,k}, \mu_{j,k}, \sigma_{j,k}, \rho_{j,k}$.

Table 3 below shows risk amounts, that is $risk_{j,k}^{RC}$ and $risk_{j,k}^{EC}$ (RC and EC in the table, respectively) of each segment within each BU, in addition to the exposure $A_{j,k}$, the risk parameters $p_{j,k}, \mu_{j,k}, \sigma_{j,k}, \rho_{j,k}$ and the profit rate $p\_rate_{j,k}$. Particularly, $risk_{j,k}^{RC}$ of industrial sectors (j) $(j = 1, \ldots, 12)$ are arranged in descending order for each BU (k) $(k = 1, 2)$, where both $risk_{j,k}^{RC}$ and $risk_{j,k}^{EC}$ depend on the exposure $A_{j,k}$ as well as parameters $p_{j,k}, \mu_{j,k}, \rho_{j,k}$, β_k, and $risk_{j,k}^{EC}$ also depends on $\sigma_{j,k}$.

The last line for each BU(k) $(k = 1, 2)$ shows the weighted average value of each $p_{j,k}$, $\mu_{j,k}$, $\sigma_{j,k}$, $\rho_{j,k}$, $p\_rate_{j,k}$ and the subtotal of each $A_{j,k}$, $risk_{j,k}^{RC\_IRB}$, $risk_{j,k}^{RC}$, $risk_{j,k}^{EC}$, where the subtotals of $risk_{j,k}^{RC}$ and $risk_{j,k}^{EC}$ are equal for each BU(k) $(k = 1, 2)$.

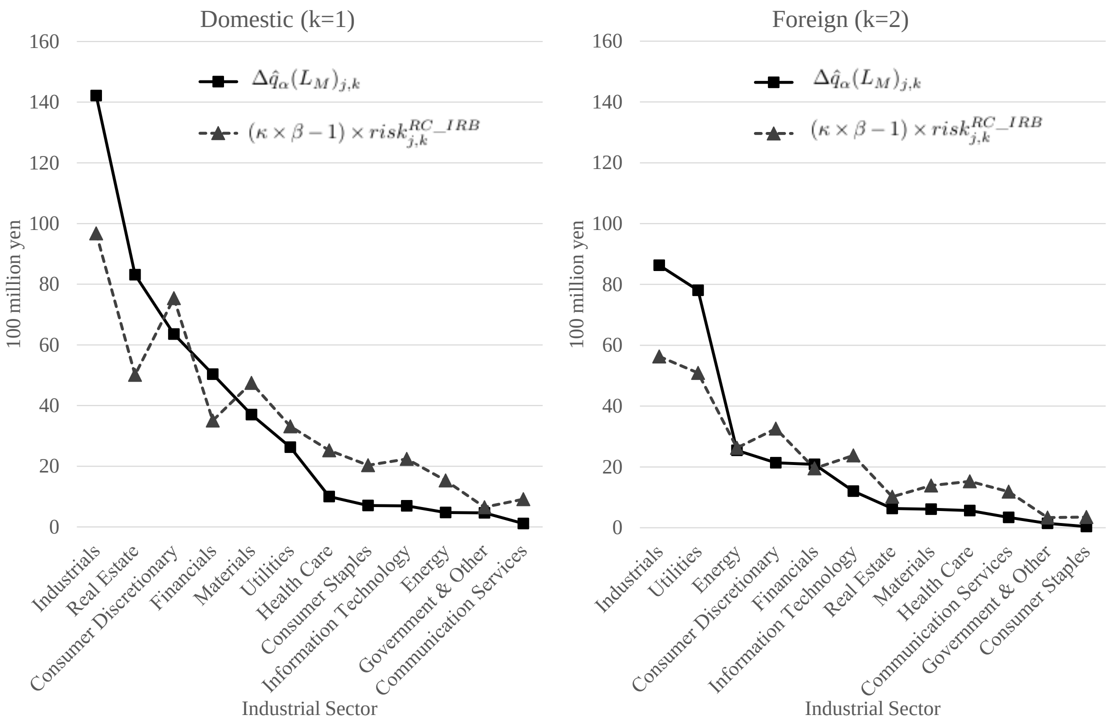
We observe that $risk_{j,k}^{EC}$ is smaller than $risk_{j,k}^{RC}$ for all segments included in the bottom 6 for each BU. Comparing (2.17) and (2.18), this implies that the approximate granularity adjustment term, (2.19) is smaller than $(\kappa \times \beta_k - 1) \times risk_{j,k}^{RC\_IRB}$ given $\kappa \times \beta_k > 1$ in each segment for the bottom 6 of each BU, which is however, not the case for each segment in the top 6 segments of each BU.

Table3 Risk and profit parameters by segment $\gamma(j,k)$: $A_{j,k}$: 100 million yen, $risk_{j,k}^{RC}$: (base case: $\beta_1 = 1.6276$ and $\beta_2 = 1.5586$, 100 million yen), $risk_{j,k}^{EC}$: (25% concentration, 100 million yen), Abbreviations: RC=$risk_{j,k}^{RC}$, EC= $risk_{j,k}^{EC}$.

BU (k)	Industrial sector (j)	$A_{j,k}$	$p_{j,k}$	$\mu_{j,k}$	$\sigma_{j,k}$	$\rho_{j,k}$	$p\_rate_{j,k}$	RC	EC
Domestic (1)	Industrials (1)	12,000	1.06%	25%	25%	44%	1.31%	625	670
Domestic (1)	Consumer Discretionary (2)	8,000	1.28%	27%	25%	43%	1.44%	487	475
Domestic (1)	Real Estate (3)	9,000	0.40%	19%	25%	47%	0.83%	324	357
Domestic (1)	Materials (4)	6,000	1.00%	25%	25%	44%	1.26%	307	296
Domestic (1)	Financials (5)	7,000	0.31%	25%	25%	47%	0.74%	227	242
Domestic (1)	Utilities (6)	5,000	0.22%	29%	25%	48%	0.66%	214	207
Domestic (1)	Health Care (7)	3,000	0.70%	30%	25%	45%	1.00%	163	148
Domestic (1)	Information Technology (8)	2,500	0.92%	29%	25%	44%	1.17%	145	129
Domestic (1)	Consumer Staples (9)	2,500	1.21%	24%	25%	43%	1.43%	131	118
Domestic (1)	Energy (10)	2,000	1.70%	20%	25%	41%	1.86%	99	89
Domestic (1)	Communication Services (11)	1,000	1.21%	27%	25%	43%	1.40%	59	51
Domestic (1)	Government & Other (12)	2,000	0.14%	25%	25%	48%	0.61%	42	40
Domestic (1)	Subtotal/Weighted average	60,000	0.80%	25%	25%	45%	1.11%	2,823	2,823
Foreign (2)	Industrials (1)	8,000	2.12%	25%	31%	40%	2.44%	487	517
Foreign (2)	Utilities (6)	8,000	0.43%	29%	31%	47%	1.14%	440	468
Foreign (2)	Consumer Discretionary (2)	4,000	2.56%	27%	31%	39%	2.71%	282	270
Foreign (2)	Energy (10)	4,000	3.40%	20%	31%	38%	3.55%	227	226
Foreign (2)	Information Technology (8)	3,000	1.84%	29%	31%	41%	2.15%	206	194
Foreign (2)	Financials (5)	4,000	0.62%	25%	31%	46%	1.30%	169	170
Foreign (2)	Health Care (7)	2,000	1.39%	30%	31%	42%	1.81%	132	122
Foreign (2)	Materials (4)	2,000	2.00%	25%	31%	41%	2.35%	120	112
Foreign (2)	Communication Services (11)	1,500	2.42%	27%	31%	39%	2.61%	103	94
Foreign (2)	Real Estate (3)	2,000	0.79%	19%	31%	45%	1.48%	88	84
Foreign (2)	Consumer Staples (9)	500	2.42%	24%	31%	39%	2.68%	30	27
Foreign (2)	Government & Other (12)	1,000	0.27%	25%	31%	47%	1.04%	29	27
Foreign (2)	Subtotal/Weighted average	40,000	1.64%	26%	31%	45%	2.07%	2,312	2,312
	Total/Weighted average	100,000	1.14%	25%	27%	44%	1.50%	5,135	5,135

We remark that as the exposure $A_{j,k}$ decreases (increases), the approximation of the granularity adjustment term (2.19) decreases (increases) in a nonlinear way as the line of $\Delta\hat{q}_\alpha(L_M)_{j,k}$ in the next Figure 1, which seems the main reason for small $risk^{EC}_{j,k}$ relative to $risk^{RC}_{j,k}$ for all segments in the bottom 6 for each BU.

Figure1



On the contrary, the term (2.19) may work as an adjustment of over-approximation by $q_\alpha(E[L_M|X])$ for $q_\alpha(L_M)$ in (3.2) against changes in the parameters $p_{j,k}, \mu_{j,k}, \rho_{j,k}$, which has more impacts than increase in the exposure $A_{j,k}$ to make $risk^{RC}_{j,k}$ larger than $risk^{EC}_{j,k}$ for some cases in the top 6 segments.

$$q_\alpha(L_M) = q_\alpha(E[L_M|X]) + \Delta q_\alpha(L_M) \simeq q_\alpha(E[L_M|X]) + \Delta\hat{q}_\alpha(L_M) \qquad (3.2)$$

We also note that since $\sigma_{j,k}$ appears only in (2.19), the increase (decrease) in the parameter $\sigma_{j,k}$ does not affect $risk^{RC}_{j,k}$, but makes $risk^{EC}_{j,k}$ increase (decrease).

Next, Table 4 below shows two cases a and b of the exposure allocations within segment $\gamma(j,k)$, which are prepared for comparison of the approximate granularity adjustment term $\Delta\hat{q}_\alpha(L_M)$ (and hence, $risk^{EC}$) by the difference in the exposure allocation. The exposure allocation case b, that is 25% concentrated on one obligor in a segment, is typical for international banks.

Then, for six cases of different number of obligors M (240, 2,400, 24,000) and exposure allocations within segment $\gamma(j,k)$ (a,b in Table 4), Table 5 below shows the calculation result for $risk^{RC\_IRB}$, $risk^{RC}$, $\Delta\hat{q}_\alpha(L_M)$ and $risk^{EC}$ based on the equations (2.11), (2.14), (2.7) and (2.15), respectively, where we assume α=99.9%, and set 1 year holding period, β_1=1.6276, β_2=1.5586 and κ=0.725, with β_1 and β_2 (in stead of β) representing domestic and foreign β_k, respectively. We will explain how to set β_1, β_2 and κ in detail at the end of Section 3.3.

As for the number of obligors, since we set two BUs ($n = 2$) and twelve industrial sectors ($m = 12$), we have 24 obligors as the minimum, based on which we calculate the credit risk amounts for greater number of obligors namely 2,400 and 24,000. We remark that a Japanese major bank has the same order of obilgors as 24,000.

Moreover, we can see in Table 5 that $\Delta\hat{q}_\alpha(L_M)$ decreases as the number of obligors M increases. In addition, for the case of 25% concentration, where the exposure is concentrated on one obligor in a segment, there is few differences in $\Delta\hat{q}_\alpha(L_M)$ between 24,000 and 2,400 obligors. Therefore, we use 2,400 obligors for portfolio optimization to reduce the computational burden without loss of reality. The details of the exposure allocation within segments $\gamma(j,k)$ are shown in Appendix.A.

In the portfolio optimization results in Section 3.4 and the stress test for the optimized portfolio in Section 3.5, only the case b (25% concentration) in Table 4 is used. This is because the equal allocation in the case a is not a realistic setting for typical loan portfolios of international banks, though it is necessary for comparing the approximate granularity adjustment terms in Table 5.

Table4 Exposure allocation within segment $\gamma(j,k)$

Case	Type	Description
a	Equal allocation	The loan exposure is equally allocated to all obligors
b	25% concentration	25% of the exposure is concentrated on one obligor, and the rest is equally allocated

Table5

Case	Exposure allocation type	Number of obligors M	$risk^{RC\_IRB}$	$risk^{RC}$	$\Delta\hat{q}_\alpha(L_M)$	$risk^{EC}$
a-1	Equal allocation	240	4,431	5,135	772	5,203
a-2	Equal allocation	2,400	4,431	5,135	77	4,508
a-3	Equal allocation	24,000	4,431	5,135	8	4,439
b-1	25% concentration	240	4,431	5,135	1,143	5,574
b-2	25% concentration	2,400	4,431	5,135	704	5,135
b-3	25% concentration	24,000	4,431	5,135	665	5,096

3.3 Constraints for optimization

To obtain a capital allocation that achieves the maximum profit, we prepare four constraints as explained in Problem 2.1. First, let us recall that $RiskCapasity$ stands for the regulatory required capital, $RiskAppetite_k$ for the BU k's risk appetite, and $RiskLimit$ for the upper risk limit applied to each segment (j,k) (industrial sector j and BU k). Also, the exposure change ratio δ denotes a loan market constraint. Next, reflecting the current situation of major banks, we explain how to set these four constraints in detail from a practical viewpoint as follows:

1. $RiskCapacity$: $RiskCapacity$ is set as $1.125 \times risk^{RC}$, because we prepare 12.5% of the capital buffer for regulatory capital credit risk $risk^{RC}$, which can be divided into four items: As the regulatory requirement, the first three items are capital conservation buffer (2.5%), countercyclical buffer (0.0%-2.5%), and global systematically important banks (G-SIBs) buffer (1.0%-2.5%). The last item in our framework is the other business buffer mainly for foreign exchange rate fluctuations and model parameter updates, which is assumed to be 5.0%, since Japanese international banks maintain an additional capital buffer of around 5%. Therefore, since $risk^{RC} = 5,135$ in Table 5, $RiskCapacity$ is set with an approximation as follows:

$RiskCapacity = risk^{RC} \times 1.125$ =5,135× 1.125=5,777 $\simeq$ 5,800.

Here, we suppose that our hypothetical bank with the loan portfolio in Table 2 initially has at least 5,800 capital amount, since in practice a financial institution should set its RiskCapacity as the amount smaller than actual regulatory capital owned by the financial institution.

2. $RiskAppetite_k$: Following a conventional way of Japanese megabanks, we set $RiskAppetite_k$ to be $RiskCapacity$ $\times$ (k's exposure ratio) with some adjustment specific to the business unit k, where the k's exposure ratio is determined based on the current exposure of a typical international bank.

   In this numerical analysis, according to the subtotal of $A_{j,k}$ for Domestic (1) and Foreign (2) in Table 2, we use $RiskCapacity = 5,800$ above with exposure ratio 0.6 and 0.4 for the domestic ($k = 1$) and foreign ($k = 2$) obligors, respectively, as well as with downward adjustment for $k = 1$ and upward adjustment for $k = 2$. Then, we have 3,400 for $RiskAppetite_1$ and 2,400 for $RiskAppetite_2$.

   However, a more reasonable way of the allocation is the one based on the risk amounts, namely $risk_1^{RC} = 2,823$ and $risk_2^{RC} = 2,312$ for the subtotals of Domestic(1) and Foreign (2), respectively in Table 2. Further, since the foreign unit is more profitable, we will show an example in Section 3.4.1, where $RiskCapacity$ is allocated to the domestic and foreign units in a 5 to 5 ratio, which is a little bit more to the foreign unit than the ratio 5:4 based on the risk amounts.

3. $RiskLimit$: According to a survey by IACPM (International Association of Credit Portfolio Managers) and a hearing survey one of the authors could obtain, lending concentration on one segment is around 10% of the regulatory risk capital ($RiskCapacity$) in international major banks. However, since we have smaller numbers of segments than actual due to the limitation of available data, we allow more lending concentration up to 12.5% of $RiskCapacity$ in our numerical analysis. Hence, $RiskLimit$ is calculated as 5,800 $\times$ 0.125 = 725.

   In addition, we note that each $RiskAppetite_k$ is set to be a lower level than $RiskLimit$. Concretely, $RiskAppetite$ are set as $RiskAppetite_1 = 3,400$ and $RiskAppetite_2 = 2,400$ for 12 segments in the domestic and foreign business units, respectively, while total $RiskLimit$ for each unit is set as 8,700(= 725 $\times$ 12).

4. The exposure change ratio δ: Firstly, let us recall that we compute a portfolio which maximizes profits by changing the exposure on the top six risk segments with respect to the regulatory capital credit risk $risk_{j,k}^{RC}$ for each domestic and foreign BU, namely, the segments ranked from the 1st to 6th in terns of $risk_{j,k}^{RC}$ shown in Table 3.

   Then, two types of the ratio δ are prepared, where the two δ are set with reference to "the annual growth rate of loans to large corporations" in the investor briefing materials [15],[19],[14].

   Particularly, as a standard level of the ratio δ, δ is set as 3% corresponding to the pre-COVID-19 crisis period, which is also realistic in adjustment of loan amounts during the next 1 year. On the contrary, δ is set as 20%, a maximum level of δ corresponding to the COVID-19 crisis period, whence obligors had much difficulty in the use of the other funding instruments such as issuance of corporate bonds. In addition, setting $\delta = 20\%$ is more useful than 3% to highlight the characteristics of the optimal portfolio.

   In addition, we apply $\delta = 0\%$ in optimization against the segments ranked from the 7th to 12th for each $k = 1, 2$ in terms of $risk_{j,k}^{RC}$ shown in Table 3, because it seems almost impossible to raise the loan amounts for those segments even when the profit rates $p\_rate_{j,k}$ in Table 3 look high, due to lack of demand for domestic customers and lack of competitiveness for foreign customers in terms of the lending rates in the loan markets. Also, a decrease in the loan from such lower ranked segments is unrealistic within 1 year.

The amount of each constraint in (2.26) is summarized in Table 6:

Table6 Constraint amount

Constraint	Amount (100 million yen)
$RiskCapacity$	5,800
$RiskAppetite_1$	$3,400$
$RiskAppetite_2$	$2,400$
$RiskLimit$	725
The exposure change ratio δ	3% or 20%

When optimizing the portfolio, we change the exposure $A_{j,k}$ only for the top 6 risk segments with respect to $risk^{RC}$ in each BU: Namely, in the domestic BU, we change the exposures of Industrials, Consumer Discretionary, Real Estate, Financials, Materials, and Utilities. In the foreign BU, we change the exposures of Industrials, Consumer Discretionary, Financials, Utilities, Energy, and Information Technology.

Next, let us explain the reason for assuming $\beta_1 = 1.6272$ and $\beta_2 = 1.5586$ in Table 3 and recall that this coefficient β_k is used to obtain $risk^{RC\_SA}$ for each business unit based on the equation (2.13), namely, $risk^{RC\_SA} = \beta_k \times risk^{RC\_IRB}$.

As explained at the beginning of the paragraph in Section 2.2, the regulatory capital credit risk ($risk^{RC}$) is the regulatory capital in our framework, and also, the minimum amount of the regulatory capital must be 8% of risk-weighted assets (RWAs), where RWAs are calculated by the Standardized Approach (SA) or Internal Rating Based Approach (IRB). Therefore, we have $risk^{RC} = 0.08\times$ risk-weighted assets (RWAs).

According to Table 1 in [4], "Credit risk RWAs" are given as 800 for RWAs based on internal models (IRWA) and 1,150 for RWAs based on standardized Approaches (SRWA). Then, β is determined by β= 1.4375: Namely, using the equation (2.13) with $risk^{RC\_SA} = (0.08\times$ SRWA) and $risk^{RC\_IRB} = (0.08\times$ IRWA), we calculate β as 1.4375 = (0.08 $\times$ SRWA)/(0.08$\times$ IRWA)= 1,150/800.

However, following the revision of Chapter 7 in Kanemoto[11], the level of 1,150 will be increased. Further, loan portfolio of the domestic BU is more concentrated than that of the foreign BU in a Japanese major bank. Thus, we prepare two cases of β_1 and β_2, with β_1 and β_2 representing domestic and foreign β_k, respectively.

In particular, as a base case, we set $\beta_1 = 1.6276$ and $\beta_2 = 1.5586$, where we use the ratio of $risk^{RC\_SA}$ computed by the standardized approach (SA) with external rating relative to $risk^{RC\_IRB}$ calculated by the internal rating approach (IRB) for a recent corporate loan portfolio of a Japanese megabank. As a more conservative case, we raise those to $\beta_1 = 1.6966$ and $\beta_2 = 1.6276$ by reflecting that RWAs based on SA will become increased by following the revision of Chapter 7 in Kanemoto[11].

Moreover, Table 4 in "Basel III Monitoring Report: Results of the cumulative quantitative impact study, December 2017" informs that the ratio of the final IRB risk weight relative to the final SA risk weight without capital floor is 0.60 in terms of the weighted average for the total asset class, which provides an estimate of β as about 1.666($\risingdotseq$ 1/0.6) to justify our above assumption for its level within the range of 1.5 to 1.7.

Finally, let us explain the reason for assuming κ to be $\kappa = 0.725$. According to p.7 of [5], "banks' capital requirements do not fall below 72.5% of the RWAs computed by the SA," which is scheduled to be adopted by January 1st, 2028. This means in our context that the regulatory capital credit risk $risk^{RC}$ is not allowed to fall below 72.5% of $risk^{RC\_SA}$. Therefore, we set κ

in (2.17) as κ=0.725.

3.4 Portfolio optimization results

This subsection presents and explains the solutions to Problem 2.1 in Section 2.3.5 by the Excel's GRG (Generalized Reduced Gradient) solver with parameters and constraints provided in the previous subsections.

First, let us recall that for $k = 1, 2$, the ratio of $risk_k^{RC}$ relative to $risk_k^{RC\_IRB}$ is given as $\kappa \times \beta_k$ with $\kappa = 0.725$. Then, in the base case with $\beta_1 = 1.6276$ and $\beta_2 = 1.5586$, the ratios $\kappa \times \beta_k$ are given as 1.18 for the domestic unit ($k = 1$) and 1.13 for the foreign unit ($k = 2$) , which are the same as ratios of $risk_k^{EC}$($risk_k^{RC\_IRB}$ plus the approximate granularity adjustment) relative to $risk_k^{RC\_IRB}$. Hence, this case may highlight the difference of the effects in optimization between the approximate granularity adjustment in $risk_k^{EC}$, i.e. $\Delta\hat{q}_\alpha(L_M)$ in (3.2) and the increase of $risk_k^{RC}$ from $risk_k^{RC\_IRB}$, i.e., $(\kappa \times \beta_k - 1) \times risk^{RC\_IRB}$ in (3.3).

$$risk^{RC} = risk^{RC\_IRB} + (\kappa \times \beta_k - 1) \times risk^{RC\_IRB}, \tag{3.3}$$

On the contrary, in the conservative case with $\beta_1 = 1.6966$ and $\beta_2 = 1.6276$, the ratios $\kappa \times \beta_k$ are given as 1.23 for the domestic unit and 1.18 for the foreign unit, which are higher than the ratios of $risk_k^{EC}$ relative to $risk_k^{RC\_IRB}$ that are given by 1.18 for $k = 1$ and 1.13 for $k = 2$. That is, $risk_k^{RC}$ with conservative β evaluates the risks more stringent than $risk_k^{EC}$.

We start with a normal loan market constraint, i.e. $\delta = 3\%$. Firstly, in Table 7 below with base case β, the profit optimization obviously raises the $\tilde{profit}$ as well as $\tilde{risk}^{RC}$, but the results for all cases I, II and III are the same in $\tilde{profit}$ and $\tilde{risk}^{RC}$, since as observed in Table 8, the loan market constraint $\delta = 3\%$ is attained for all the segments before the other constraints are effective.

In detail, Table 8 shows that in the initial state (named as "Initial"), each total $risk^{RC}$ and $risk^{EC}$ is given by 5,135 against the constraint 5,800, and that each BU $risk_k^{RC}$ and $risk_k^{EC}$ is 2,823 for the domestic unit($k = 1$) against the constraint 3,400, while 2,312 for the foreign unit($k = 2$) against the constraint 2,400. Each segment $risk_{j,k}^{RC}$ and $risk_{j,k}^{EC}$ is within the constraint 725, where each maximum is $risk_{1,1}^{RC} = 625$ and $risk_{1,1}^{EC} = 670$ for the segment 'Industrials' in the domestic unit ($k = 1$).

Then, the exposures become increased up to the loan market constrained $\delta = 3\%$ for all the segments in each case I, II and III.

Table7 Portfolio optimization result: normal loan market constraint $\delta = 3\%$; base case: $\beta_1 = 1.6276$ and $\beta_2 = 1.5586$; $RiskCapacity = 5,800$ (100 million yen).

	Initial			Optimized			Optimized-Initial		
Case	I	II	III	I	II	III	I	II	III
$\tilde{risk}^{RC}$ (100 million yen)	5,135	5,135	5,135	5,255	5,255	5,255	120	120	120
$\tilde{profit}$ (100 million yen)	1,496	1,496	1,496	1,531	1,531	1,531	35	35	35

Table8 Optimized exposure under constraints: normal loan market constraint $\delta = 3\%$, $RiskAppetite_1 = 3,400$, $RiskAppetite_2 = 2,400$; $risk^{RC}_{j,k}$: (base case: $\beta_1 = 1.6276$ and $\beta_2 = 1.5586$, 100 million yen), $risk^{EC}_{j,k}$: (25% concentration, 100 million yen), $A_{j,k}$: initial exposure (100 million yen); $\tilde{A}_{j,k}$:optimized exposure (100 million yen): The check mark ✓ indicates that the constraint δ is reached. p_rate_k and p_rate are weighted averages of $p\_rate_{j,k}$ over $j = 1, \ldots, 12$, and $k = 1, 2$, $j = 1, \ldots, 12$, respectively.

		Initial				Optimized I		Optimized II		Optimized III	
BU (k)	Industrial sector (j)	$A_{j,k}$	$p\_rate_{j,k}$	$risk^{RC}_{j,k}$	$risk^{EC}_{j,k}$	$\tilde{A}_{j,k}$	δ	$\tilde{A}_{j,k}$	δ	$\tilde{A}_{j,k}$	δ
Domestic (1)	Industrials (1)	12,000	1.31%	625	670	12,360	✓	12,360	✓	12,360	✓
Domestic (1)	Consumer Discretionary (2)	8,000	1.44%	487	475	8,240	✓	8,240	✓	8,240	✓
Domestic (1)	Real Estate (3)	9,000	0.83%	324	357	9,270	✓	9,270	✓	9,270	✓
Domestic (1)	Materials (4)	6,000	1.26%	307	296	6,180	✓	6,180	✓	6,180	✓
Domestic (1)	Financials (5)	7,000	0.74%	227	242	7,210	✓	7,210	✓	7,210	✓
Domestic (1)	Utilities (6)	5,000	0.66%	214	207	5,150	✓	5,150	✓	5,150	✓
Foreign (2)	Industrials (1)	8,000	2.44%	487	517	8,240	✓	8,240	✓	8,240	✓
Foreign (2)	Utilities (6)	8,000	1.14%	440	468	8,240	✓	8,240	✓	8,240	✓
Foreign (2)	Consumer Discretionary (2)	4,000	2.71%	282	270	4,120	✓	4,120	✓	4,120	✓
Foreign (2)	Energy (10)	4,000	3.55%	227	226	4,120	✓	4,120	✓	4,120	✓
Foreign (2)	Information Technology (8)	3,000	2.15%	206	194	3,090	✓	3,090	✓	3,090	✓
Foreign (2)	Financials (5)	4,000	1.30%	169	170	4,120	✓	4,120	✓	4,120	✓
BU (k)	Industrial sector (j)	A_k	p_rate_k	$risk^{RC}_k$	$risk^{EC}_k$	$\tilde{A}_k$	-	$\tilde{A}_k$	-	$\tilde{A}_k$	-
Domestic (1)	Subtotal ($j = 1, \ldots, 12$)	60,000	1.11%	2,823	2,823	61,410		61,410		61,410	
Foreign (2)	Subtotal ($j = 1, \ldots, 12$)	40,000	2.07%	2,312	2,312	40,930		40,930		40,930	
BU (k)	Industrial sector (j)	A	p_rate	$risk^{RC}$	$risk^{EC}$	$\tilde{A}$	-	$\tilde{A}$	-	$\tilde{A}$	-
Total ($k = 1, 2$)	Total ($j = 1, \ldots, 12$)	100,000	1.50%	5,135	5,135	102,340		102,340		102,340	

In the next Table 9, the optimized $\tilde{profit}$ become smaller with larger $\tilde{risk}^{RC}$ than those in base case β in Table 7 for case I and II due to the conservative β, which is used in calculation of both $\tilde{risk}^{RC}_k$ and $\tilde{risk}^{RC}_{j,k}$ for case I, and $\tilde{risk}^{RC}_k$ for case II. (In case II, $\tilde{risk}^{EC}_{j,k}$ is applied in the segment risk.) We also remark that as the foreign unit's risk amount $\tilde{risk}^{RC}_2$ is 2,405 in the initial state('Initial'), which already exceeds its constraint, $RiskAppetite_2 = 2,400$, the risk should be reduced in profit optimization.

In detail, for those cases, Table 10 shows that after optimization, $RiskAppetite_2$=2,400 for the foreign BU is reached before the constraint for the increase in exposures (δ=3%) is reached for two segments(Information Technologies and Financials) in the foreign BU. It also shows that the constraint for the decrease in exposures ($-\delta = -3\%$) is reached against two segments(Industrials and Utilities) in the foreign BU.

On the contrary, for case III the optimized $\tilde{profit}$ is unchanged from the one in Table 7, and becomes increased from those for I and II in Table 9, thanks to $\tilde{risk}^{EC}_k$ used for each business unit ($k = 1, 2$) in case III, in addition to $\tilde{risk}^{EC}_{j,k}$ used for each segment ($j = 1, \ldots, 12$, $k = 1, 2$). In particular, Table 10 indicates that less stringent risk evaluation in $\tilde{risk}^{EC}_k$ than in $\tilde{risk}^{RC}_k$ makes $RiskAppetite_2$=2,400 not reached for both BUs to expand the credit exposures to all the segments up to the upper limit δ=3%.

Consequently, the profit in case III using $\tilde{risk}^{EC}$ for both BU and segment constraints ($RiskAppetite_k$ and $RiskLimit$) becomes larger than the other cases by 1,700 million yen. We note that since Japanese megabanks hold around 10 times exposures as much as in the current example, the difference in profits is not negligible at all.

Table9 Portfolio optimization result: normal loan market constraint $\delta = 3\%$; conservative case: $\beta_1 = 1.6966$ and $\beta_2 = 1.6276$; $RiskCapacity = 5,800$ (100 million yen).

	Initial			Optimized			Optimized-Initial		
Case	I	II	III	I	II	III	I	II	III
$\tilde{risk}^{RC}$ (100 million yen)	5,341	5,341	5,341	5,404	5,404	5,466	63	63	125
$\tilde{profit}$ (100 million yen)	1,496	1,496	1,496	1,514	1,514	1,531	18	18	35

Table10 Optimized exposure under constraints: normal loan market constraint $\delta = 3\%$, $RiskAppetite_1 = 3,400$, $RiskAppetite_2 = 2,400$; $risk^{RC}_{j,k}$: (conservative case: $\beta_1 = 1.6966$ and $\beta_2 = 1.6276$, 100 million yen), $risk^{EC}_{j,k}$: (25% concentration, 100 million yen), $A_{j,k}$: initial exposure (100 million yen); $\tilde{A}_{j,k}$:optimized exposure (100 million yen): The check mark ✓ indicates that the constraint δ is reached. The *bold* with * font indicates that the constraint amounts of *RiskAppetite* are reached, respectively. p_rate_k and p_rate are weighted averages of $p\_rate_{j,k}$ over $j = 1, \ldots, 12$, and $k = 1, 2$, $j = 1, \ldots, 12$, respectively.

		Initial				Optimized I		Optimized II		Optimized III	
BU (k)	Industrial sector (j)	$A_{j,k}$	$p\_rate_{j,k}$	$risk^{RC}_{j,k}$	$risk^{EC}_{j,k}$	$\tilde{A}_{j,k}$	δ	$\tilde{A}_{j,k}$	δ	$\tilde{A}_{j,k}$	δ
Domestic (1)	Industrials (1)	12,000	1.31%	650	670	12,360	✓	12,360	✓	12,360	✓
Domestic (1)	Consumer Discretionary (2)	8,000	1.44%	507	475	8,240	✓	8,240	✓	8,240	✓
Domestic (1)	Real Estate (3)	9,000	0.83%	337	357	9,270	✓	9,270	✓	9,270	✓
Domestic (1)	Materials (4)	6,000	1.26%	319	296	6,180	✓	6,180	✓	6,180	✓
Domestic (1)	Financials (5)	7,000	0.74%	236	242	7,210	✓	7,210	✓	7,210	✓
Domestic (1)	Utilities (6)	5,000	0.66%	223	207	5,150	✓	5,150	✓	5,150	✓
Foreign (2)	Industrials (1)	8,000	2.44%	507	517	8,240	✓	8,240	✓	8,240	✓
Foreign (2)	Utilities (6)	8,000	1.14%	458	468	7,760	✓	7,760	✓	8,240	✓
Foreign (2)	Consumer Discretionary (2)	4,000	2.71%	293	270	3,975		3,975		4,120	✓
Foreign (2)	Energy (10)	4,000	3.55%	236	226	4,120	✓	4,120	✓	4,120	✓
Foreign (2)	Information Technology (8)	3,000	2.15%	214	194	2,910	✓	2,910	✓	3,090	✓
Foreign (2)	Financials (5)	4,000	1.30%	175	170	3,880	✓	3,880	✓	4,120	✓
BU (k)	Industrial sector (j)	A_k	p_rate_k	$risk^{RC}_k$	$risk^{EC}_k$	$\tilde{A}_k$	-	$\tilde{A}_k$	-	$\tilde{A}_k$	-
Domestic (1)	Subtotal ($j = 1, \ldots, 12$)	60,000	1.11%	2,936	2,823	61,410		61,410		61,410	
Foreign (2)	Subtotal ($j = 1, \ldots, 12$)	40,000	2.07%	2,405	2,312	**39,885***		**39,885***		40,930	
BU (k)	Industrial sector (j)	A	p_rate	$risk^{RC}$	$risk^{EC}$	$\tilde{A}$	-	$\tilde{A}$	-	$\tilde{A}$	-
Total ($k = 1, 2$)	Total ($j = 1, \ldots, 12$)	100,000	1.50%	5,341	5,135	101,295		101,295		102,340	

The following Table 11 – 14 show the case for an expanded loan market constraint, $\delta = 20\%$,which occurs when direct financing such as bond issues becomes difficult for obligors as in COVID-19. First, we observe that the larger $\delta = 20\%$ makes its optimized $\tilde{profit}$ more with $\tilde{risk}^{RC}$ closer to its limit $RiskCapacity = 5,800$ than in Table 7 and 9 with $\delta = 3\%$. Also, thanks to less conservative β, we see the larger optimized $\tilde{profit}$ with smaller $\tilde{risk}^{RC}$ in Table 11 than in Table 13.

Moreover, in Table 11, the largest optimized $\tilde{profit}$ is given in case I, the next in case III and the smallest in case II: Let us recall that as can be seen from Figure 1, $risk^{EC}$ including the approximate granularity adjustment takes more concentration risks on the largest exposures into account than $risk^{RC}$, especially in the domestic unit, which makes $RiskLimit = 725$ reached faster for 'Industrials' ($j = 1, k = 1$), that is the largest $risk^{EC}_{j,k}$ domestic segment. As a result, exposures $\tilde{A}_{1,1}$ in case II and III become smaller than in case I after optimization, as shown in Table 12, and hence the optimized $\tilde{profit}$s in cases II and III with $risk^{EC}_{j,k}$ are less than in case I with $risk^{RC}_{j,k}$.

We also note that since the approximate granularity adjustment evaluates less concentration risks on all the bottom 6 segments' small exposures than $(\kappa \times \beta - 1) \times risk^{RC\_IRB}_{j,k}$ as observed in Figure 1, the case III using $risk^{EC}_k$ for the BU risk is able to utilize such saved risks to increase exposures for the top 6 segments. Thus, the optimized $\tilde{profit}$ in case III could be larger than in case II with $risk^{RC}_k$ ($k = 1, 2$) for the BU constraint. Particularly, in the current case Table

12 shows that aggregate exposures to the foreign unit in case III become larger than in case II to make more profits: Namely, the sum of $\tilde{A}_{j,2}$ over $j = 1, \ldots, 12$ is 41,106 in case III, while 40,948 in case II.

Finally, we observe that in Table 13, the largest optimized $\widetilde{profit}$ is given in case III with $\widetilde{risk}^{RC}$ reached at the constraint $RiskCapacity = 5,800$, the next in case I and the smallest in case II: The $\widetilde{profit}$ in Case I becomes smaller than in case III in exchange for more stringent risk assessments due to the conservative β_k used in calculation of all the risks.

Table11 Portfolio optimization result: expanded loan market constraint $\delta = 20\%$; base case: $\beta_1 = 1.6276$ and $\beta_2 = 1.5586$; $RiskCapacity = 5,800$ (100 million yen).

	Initial			Optimized			Optimized-Initial		
Case	I	II	III	I	II	III	I	II	III
$\widetilde{risk}^{RC}$ (100 million yen)	5,135	5,135	5,135	5,635	5,586	5,598	499	451	462
$\widetilde{profit}$ (100 million yen)	1,496	1,496	1,496	1,655	1,643	1,646	159	147	150

Table12 Optimized exposure under constraints: expanded loan market constraint $\delta = 20\%$, $RiskAppetite_1 = 3,400$, $RiskAppetite_2 = 2,400$; $risk^{RC}_{j,k}$: (base case: $\beta_1 = 1.6276$ and $\beta_2 = 1.5586$, 100 million yen), $risk^{EC}_{j,k}$: (25% concentration, 100 million yen), $A_{j,k}$: initial exposure (100 million yen); $\tilde{A}_{j,k}$:optimized exposure (100 million yen): The check mark ✓ indicates that the constraint δ is reached. The *bold* and *bold* with * fonts indicate that the constraint amounts of *RiskLimit* and *RiskAppetite* are reached, respectively. p_rate_k and p_rate are weighted averages of $p\_rate_{j,k}$ over $j = 1, \ldots, 12$, and $k = 1, 2$, $j = 1, \ldots, 12$, respectively.

		Initial				Optimized I		Optimized II		Optimized III	
BU (k)	Industrial sector (j)	$A_{j,k}$	$p\_rate_{j,k}$	$risk^{RC}_{j,k}$	$risk^{EC}_{j,k}$	$\tilde{A}_{j,k}$	δ	$\tilde{A}_{j,k}$	δ	$\tilde{A}_{j,k}$	δ
Domestic (1)	Industrials (1)	12,000	1.31%	625	670	**13,919**		**12,999**		**13,001**	
Domestic (1)	Consumer Discretionary (2)	8,000	1.44%	487	475	9,600	✓	9,600	✓	9,600	✓
Domestic (1)	Real Estate (3)	9,000	0.83%	324	357	10,800	✓	10,800	✓	10,800	✓
Domestic (1)	Materials (4)	6,000	1.26%	307	296	7,200	✓	7,200	✓	7,200	✓
Domestic (1)	Financials (5)	7,000	0.74%	227	242	8,400	✓	8,400	✓	8,400	✓
Domestic (1)	Utilities (6)	5,000	0.66%	214	207	6,000	✓	6,000	✓	6,000	✓
Foreign (2)	Industrials (1)	8,000	2.44%	487	517	9,600	✓	9,600	✓	9,600	✓
Foreign (2)	Utilities (6)	8,000	1.14%	440	468	6,400	✓	6,400	✓	6,400	✓
Foreign (2)	Consumer Discretionary (2)	4,000	2.71%	282	270	4,800	✓	4,800	✓	4,800	✓
Foreign (2)	Energy (10)	4,000	3.55%	227	226	4,800	✓	4,800	✓	4,800	✓
Foreign (2)	Information Technology (8)	3,000	2.15%	206	194	3,148		3,148		3,339	
Foreign (2)	Financials (5)	4,000	1.30%	169	170	3,200	✓	3,200	✓	3,200	✓
BU (k)	Industrial sector (j)	A_k	p_rate_k	$risk^{RC}_k$	$risk^{EC}_k$	$\tilde{A}_k$	-	$\tilde{A}_k$	-	$\tilde{A}_k$	-
Domestic (1)	Subtotal ($j = 1, \ldots, 12$)	60,000	1.11%	2,823	2,823	68,919		67,999		68,001	
Foreign (2)	Subtotal ($j = 1, \ldots, 12$)	40,000	2.07%	2,312	2,312	**40,948***		**40,948***		**41,106***	
BU (k)	Industrial sector (j)	A	p_rate	$risk^{RC}$	$risk^{EC}$	$\tilde{A}$	-	$\tilde{A}$	-	$\tilde{A}$	-
Total ($k = 1, 2$)	Total ($j = 1, \ldots, 12$)	100,000	1.50%	5,135	5,135	109,868		108,948		109,107	

Table13 Portfolio optimization result: expanded loan market constraint $\delta = 20\%$; conservative case: $\beta_1 = 1.6966$ and $\beta_2 = 1.6276$; $RiskCapacity = 5,800$ (100 million yen). The *bold* font indicates that the constraint amount, *RiskCapacity* is reached.

	Initial			Optimized			Optimized-Initial		
Case	I	II	III	I	II	III	I	II	III
$\widetilde{risk}^{RC}$ (100 million yen)	5,341	5,341	5,341	5,735	5,712	**5,800**	394	371	459
$\widetilde{profit}$ (100 million yen)	1,496	1,496	1,496	1,616	1,611	1,643	120	115	147

Table14 Optimized exposure under constraints: expanded loan market constraint $\delta = 20\%$, $RiskAppetite_1 = 3,400$, $RiskAppetite_2 = 2,400$; $risk^{RC}_{j,k}$: (conservative case: $\beta_1 = 1.6966$ and $\beta_2 = 1.6276$, 100 million yen), $risk^{EC}_{j,k}$: (25% concentration, 100 million yen), $A_{j,k}$: initial exposure (100 million yen); $\tilde{A}_{j,k}$:optimized exposure (100 million yen): The check mark ✓ indicates that the constraint δ is reached. The **bold**, **bold** with * and ***bold*** with *italic* fonts indicate that the constraint amounts of $RiskLimit$, $RiskAppetite$ and $RiskCapacity$ are reached, respectively. p_rate_k and p_rate are weighted averages of $p\_rate_{j,k}$ over $j = 1, \ldots, 12$, and $k = 1, 2$, $j = 1, \ldots, 12$, respectively.

		Initial				Optimized I		Optimized II		Optimized III	
BU (k)	Industrial sector (j)	$A_{j,k}$	$p\_rate_{j,k}$	$risk^{RC}_{j,k}$	$risk^{EC}_{j,k}$	$\tilde{A}_{j,k}$	δ	$\tilde{A}_{j,k}$	δ	$\tilde{A}_{j,k}$	δ
Domestic (1)	Industrials (1)	12,000	1.31%	650	670	**13,384**		**12,966**		**12,987**	
Domestic (1)	Consumer Discretionary (2)	8,000	1.44%	507	475	9,600	✓	9,600	✓	9,600	✓
Domestic (1)	Real Estate (3)	9,000	0.83%	337	357	10,800	✓	10,800	✓	10,800	✓
Domestic (1)	Materials (4)	6,000	1.26%	319	296	7,200	✓	7,200	✓	7,200	✓
Domestic (1)	Financials (5)	7,000	0.74%	236	242	8,400	✓	8,400	✓	8,400	✓
Domestic (1)	Utilities (6)	5,000	0.66%	223	207	6,000	✓	6,000	✓	5,572	
Foreign (2)	Industrials (1)	8,000	2.44%	507	517	9,600	✓	9,600	✓	9,600	✓
Foreign (2)	Utilities (6)	8,000	1.14%	458	468	6,400	✓	6,400	✓	6,400	✓
Foreign (2)	Consumer Discretionary (2)	4,000	2.71%	293	270	4,217		4,217		4,800	✓
Foreign (2)	Energy (10)	4,000	3.55%	236	226	4,800	✓	4,800	✓	4,800	✓
Foreign (2)	Information Technology (8)	3,000	2.15%	214	194	2,400	✓	2,400	✓	3,282	
Foreign (2)	Financials (5)	4,000	1.30%	175	170	3,200	✓	3,200	✓	3,200	✓
BU (k)	Industrial sector (j)	A_k	p_rate_k	$risk^{RC}_k$	$risk^{EC}_k$	$\tilde{A}_k$	-	$\tilde{A}_k$	-	$\tilde{A}_k$	-
Domestic (1)	Subtotal ($j = 1, \ldots, 12$)	60,000	1.11%	2,936	2,823	68,384		67,966		67,559	
Foreign (2)	Subtotal ($j = 1, \ldots, 12$)	40,000	2.07%	2,405	2,312	**39,617***		**39,617***		**41,082***	
BU (k)	Industrial sector (j)	A	p_rate	$risk^{RC}$	$risk^{EC}$	$\tilde{A}$	-	$\tilde{A}$	-	$\tilde{A}$	-
Total ($k = 1, 2$)	Total ($j = 1, \ldots, 12$)	100,000	1.50%	5,341	5,135	108,001		107,583		***108,641***	

3.4.1 Case of the same risk appetite constraint for the domestic and foreign units ($RiskAppetite_1 = RiskAppetite_2$)

As explained in Section 3.3, following a conventional way of Japanese megabanks to determine the risk appetites for domestic and foreign units ($RiskAppetite_k$, $k = 1, 2$), we have so far allocated the regulatory capital-based $RiskCapacity$ to $RiskAppetite_k$, $k = 1, 2$ in a 6 to 4 ratio, based on the current exposures of a typical international bank.

However, a more reasonable way of the allocation is the one based on the risk amounts, namely around 5:4 calculated from the initial regulatory capital credit risk with the base case β, i.e., $risk^{RC}_1 = 2,823$ and $risk^{RC}_2 = 2,312$ as in Table 3. Further, since the foreign unit is more profitable, one may be willing to allocate more capital to the foreign unit.

Hence, this subsection presents a profit optimization result, where we allocate $RiskCapacity$ to $RiskAppetite_k$ ($k = 1, 2$) in a 5 to 5 ratio, which is a little bit more to the foreign unit than the ratio 5:4 based on the risk amounts. We also set the expanded loan market constraint $\delta = 20\%$ and the base case β with $\beta_1 = 1.6276$ and $\beta_1 = 1.5586$, which provides the same ratios, $risk^{RC}_k / risk^{IRB}_k$ and $risk^{EC}_k / risk^{IRB}_k$, $k = 1, 2$.

Table 15 below shows that the profits and risk amounts are the largest in case III, where $risk^{EC}$ is used in calculation of risk amounts associated with the constraints $RiskAppetite_k$ with $k = 1, 2$ and $RiskLimit$ for all k, j with $k = 1, 2$, $j = 1, \ldots, 12$.

In Table 16, we first see that the loan market constraint $\delta = 20\%$ is reached against all the optimized allocations for the foreign unit in case I, II and III. On the contrary, we observe $\tilde{A} = 60,817$ for the domestic unit (Domestic(1)) in case III, while 60,046 in Case I and 60,156 in Case II, which means that the exposure in case III becomes increased most to get the largest profits, though the constraint $RiskAppetite_1 = 2,900$ is reached for all cases. We also remark that the same allocation to all domestic segments as in case III would exceed $RiskAppetite_1$ in case II using $risk^{RC}$ for calculation of the risk amounts against the risk appetite constraint (RA), while applying $risk^{EC}$ for the risk limit constraint (RL).

Thus, even for the base case β generating the same risk evaluation between $risk_k^{RC}$ and $risk_k^{EC}$ in terms of the ratio relative to $risk_k^{IRB}$ for each $k = 1, 2$, we may conclude that it is efficient and advantageous in profit optimization to use the economic capital credit risk $risk^{EC}$ consistently against constraints for both the business units and industrial segments.

Table15 Portfolio optimization result: expanded loan market constraint $\delta = 20\%$; base case: $\beta_1 = 1.6276$ and $\beta_2 = 1.5586$; $RiskCapacity = 5,800$ (100 million yen).

	Initial			Optimized			Optimized-Initial		
Case	I	II	III	I	II	III	I	II	III
$\tilde{risk}^{RC}$ (100 million yen)	5,135	5,135	5,135	5,575	5,575	5,594	440	440	459
$\tilde{profit}$ (100 million yen)	1,496	1,496	1,496	1,651	1,651	1,655	155	155	159

Table16 Optimized exposure under constraints: expanded loan market constraint $\delta = 20\%$, $RiskAppetite_1 = 2,900$, $RiskAppetite_2 = 2,900$; $risk_{j,k}^{RC}$: (base case: $\beta_1 = 1.6276$ and $\beta_2 = 1.5586$, 100 million yen), $risk_{j,k}^{EC}$: (25% concentration, 100 million yen), $A_{j,k}$: initial exposure (100 million yen); $\tilde{A}_{j,k}$:optimized exposure (100 million yen): The check mark ✓ indicates that the constraint δ is reached. The *bold* and *bold* with * fonts indicate that the constraint amounts of $RiskLimit$ and $RiskAppetite$ are reached, respectively. p_rate_k and p_rate are weighted averages of $p\_rate_{j,k}$ over $j = 1, \ldots, 12$, and $k = 1, 2$, $j = 1, \ldots, 12$, respectively.

		Initial				Optimized I		Optimized II		Optimized III	
BU (k)	Industrial sector (j)	$A_{j,k}$	$p\_rate_{j,k}$	$risk_{j,k}^{RC}$	$risk_{j,k}^{EC}$	$\tilde{A}_{j,k}$	δ	$\tilde{A}_{j,k}$	δ	$\tilde{A}_{j,k}$	δ
Domestic (1)	Industrials (1)	12,000	1.31%	625	670	**13,919**		**12,939**		12,908	
Domestic (1)	Consumer Discretionary (2)	8,000	1.44%	487	475	9,126		9,600	✓	9,600	✓
Domestic (1)	Real Estate (3)	9,000	0.83%	324	357	7,200	✓	7,818		7,422	
Domestic (1)	Materials (4)	6,000	1.26%	307	296	7,200	✓	7,200	✓	7,200	✓
Domestic (1)	Financials (5)	7,000	0.74%	227	242	5,600	✓	5,600	✓	6,687	
Domestic (1)	Utilities (6)	5,000	0.66%	214	207	4,000	✓	4,000	✓	4,000	✓
Foreign (2)	Industrials (1)	8,000	2.44%	487	517	9,600	✓	9,600	✓	9,600	✓
Foreign (2)	Utilities (6)	8,000	1.14%	440	468	9,600	✓	9,600	✓	9,600	✓
Foreign (2)	Consumer Discretionary (2)	4,000	2.71%	282	270	4,800	✓	4,800	✓	4,800	✓
Foreign (2)	Energy (10)	4,000	3.55%	227	226	4,800	✓	4,800	✓	4,800	✓
Foreign (2)	Information Technology (8)	3,000	2.15%	206	194	3,600	✓	3,600	✓	3,600	✓
Foreign (2)	Financials (5)	4,000	1.30%	169	170	4,800	✓	4,800	✓	4,800	✓
BU (k)	Industrial sector (j)	A_k	p_rate_k	$risk_k^{RC}$	$risk_k^{EC}$	$\tilde{A}_k$	-	$\tilde{A}_k$	-	$\tilde{A}_k$	-
Domestic (1)	Subtotal ($j = 1, \ldots, 12$)	60,000	1.11%	2,823	2,823	**60,046***		**60,156***		**60,817***	
Foreign (2)	Subtotal ($j = 1, \ldots, 12$)	40,000	2.07%	2,312	2,312	46,200		46,200		46,200	
BU (k)	Industrial sector (j)	A	p_rate	$risk^{RC}$	$risk^{EC}$	$\tilde{A}$	-	$\tilde{A}$	-	$\tilde{A}$	-
Total ($k = 1, 2$)	Total ($j = 1, \ldots, 12$)	100,000	1.50%	5,135	5,135	106,246		106,356		107,017	

3.5 Stress test for the optimized portfolio

This section investigates the impact of a stressed scenario, namely sharp increases in the default probabilities $p_{j,k}$ ($j = 1, \ldots, 12$, $k = 1, 2$) of all segments on $\tilde{risk}^{RC}$ in (2.22) and $\tilde{profit}$ in (2.25). Concretely, as a stressed scenario, we raise each default probability $p_{j,k}$ to be the average during 2008-2010 that includes the global financial crisis period, while we realistically assume the base lending rates $b\_rate_{j,k}$ and the optimized portfolios to be unchanged in the following one year from those before the shock.

Table 17 shows the parameters $p_{j,k}$, $\mu_{j,k}$, $\sigma_{j,k}$ and $\rho_{j,k}$ for all the segments in the domestic and foreign units, where each $\rho_{j,k}$ is determined by the equation (3.1), while $\mu_{j,k}$ and $\sigma_{j,k}$ are the same as in Table 3 of Section 3.2.

Table17 Risk parameters by segment $\gamma(j,k)$ under the global financial crisis scenario

BU (k)	Industrial sector (j)	$p_{j,k}$	$\mu_{j,k}$	$\sigma_{j,k}$	$\rho_{j,k}$
Domestic (1)	Industrials (1)	2.21%	25%	25%	40%
Domestic (1)	Consumer Discretionary (2)	2.58%	27%	25%	39%
Domestic (1)	Real Estate (3)	1.30%	19%	25%	43%
Domestic (1)	Materials (4)	1.80%	25%	25%	41%
Domestic (1)	Financials (5)	0.71%	25%	25%	45%
Domestic (1)	Utilities (6)	0.06%	29%	25%	49%
Domestic (1)	Health Care (7)	1.29%	30%	25%	43%
Domestic (1)	Information Technology (8)	0.74%	29%	25%	45%
Domestic (1)	Consumer Staples (9)	1.56%	24%	25%	42%
Domestic (1)	Energy (10)	1.06%	20%	25%	44%
Domestic (1)	Communication Services (11)	1.56%	27%	25%	42%
Domestic (1)	Government & Other (12)	0.03%	25%	25%	49%
Foreign (2)	Industrials (1)	4.42%	25%	31%	36%
Foreign (2)	Utilities (6)	0.12%	29%	31%	48%
Foreign (2)	Consumer Discretionary (2)	5.16%	27%	31%	36%
Foreign (2)	Energy (10)	2.12%	20%	31%	40%
Foreign (2)	Information Technology (8)	1.47%	29%	31%	42%
Foreign (2)	Financials (5)	1.42%	25%	31%	42%
Foreign (2)	Health Care (7)	2.57%	30%	31%	39%
Foreign (2)	Materials (4)	3.60%	25%	31%	37%
Foreign (2)	Communication Services (11)	3.13%	27%	31%	38%
Foreign (2)	Real Estate (3)	2.61%	19%	31%	39%
Foreign (2)	Consumer Staples (9)	3.13%	24%	31%	38%
Foreign (2)	Government & Other (12)	0.06%	25%	31%	49%

Firstly, Table 18 and 19 below show $\tilde{risk}^{RC}$ and $\tilde{profit}$ with base and conservative β cases, respectively for the normal loan market constraint, i.e. $\delta = 3\%$, where the regulatory capital credit risks, $\tilde{risk}^{RC}$ in the stressed scenario (named as "Stressed" in the tables) become close to, but within the regulatory required capital $RiskCapacity = 5,800$ for all I, II, III cases. Moreover, positive $\tilde{profit}$s in the stressed scenario for all cases imply that the increase in the expected losses $\tilde{EL}$ due to the surged default probabilities could be absorbed by the expected profits.

Table18 Stress testing result: normal loan market constraint $\delta = 3\%$; base case: $\beta_1 = 1.6276$ and $\beta_2 = 1.5586$; $RiskCapacity = 5,800$ (100 million yen).

	Optimized			Stressed			Stressed-Optimized		
Case	I	II	III	I	II	III	I	II	III
$\tilde{risk}^{RC}$ (100 million yen)	5,255	5,255	5,255	5,795	5,795	5,795	540	540	540
$\tilde{profit}$ (100 million yen)	1,531	1,531	1,531	1,207	1,207	1,207	-324	-324	-324

Table19 Stress testing result: normal loan market constraint $\delta = 3\%$; conservative case: $\beta_1 = 1.6966$ and $\beta_2 = 1.6276$; $RiskCapacity = 5,800$ (100 million yen).

	Optimized			Stressed			Stressed-Optimized		
Case	I	II	III	I	II	III	I	II	III
$\tilde{risk}^{RC}$ (100 million yen)	5,402	5,402	5,466	5,743	5,743	5,795	340	340	329
$\tilde{profit}$ (100 million yen)	1,512	1,512	1,531	1,195	1,195	1,207	-317	-317	-324

Next, Table 20 and 21 show $\tilde{risk}^{RC}$ and $\tilde{profit}$ with base and conservative β cases, respectively for the expanded loan market constraint, i.e. $\delta = 20\%$,where the regulatory capital credit risks

$\tilde{risk}^{RC}$ in the stressed scenario exceed the regulatory required capital, that is $RiskCapacity = 5,800$ for all I, II, III cases. However, positive profit amounts, $\tilde{profit}$ in the stressed scenario could make up the shortage of the capital for all cases.

Table20 Stress testing result: expanded loan market constraint $\delta = 20\%$; base case: $\beta_1 = 1.6276$ and $\beta_2 = 1.5586$; $RiskCapacity = 5,800$ (100 million yen). The *bold* font indicates that the constraint amount, $RiskCapacity$ is reached.

	Optimized			Stressed			Stressed-Optimized		
Case	I	II	III	I	II	III	I	II	III
$\tilde{risk}^{RC}$ (100 million yen)	5,635	5,587	5,597	**6,303**	**6,243**	**6,253**	668	656	656
$\tilde{profit}$ (100 million yen)	1,655	1,643	1,646	1,306	1,297	1,300	-349	-346	-346

Table21 Stress testing result: expanded loan market constraint $\delta = 20\%$; conservative case: $\beta_1 = 1.6966$ and $\beta_2 = 1.6276$; $RiskCapacity = 5,800$ (100 million yen). The *bold* font indicates that the constraint amount of $RiskCapacity$ is reached.

	Optimized			Stressed			Stressed-Optimized		
Case	I	II	III	I	II	III	I	II	III
$\tilde{risk}^{RC}$ (100 million yen)	5,735	5,712	**5,800**	**6,417**	**6,389**	**6,492**	682	677	692
$\tilde{profit}$ (100 million yen)	1,616	1,611	1,643	1,276	1,272	1,296	-340	-339	-347

4 Conclusion

This paper has investigated an optimization problem for a typical loan portfolio of international banks. Specifically, considering concentration risks on large individual obligors within each industrial sector by using granularity adjustment, we have obtained a capital allocation to achieve the maximum profit for a loan portfolio under four constraints, namely the Basel regulatory capital constraint, a loan market constraint and risk limits against BUs (business units) and segments (i.e., industrial sectors within each BU). Comparing to existing research, to the best of our knowledge, this paper has analyzed the effects of different risk constraints on the optimal loan portfolio from multiple perspectives under a more realistic and comprehensive setting.

As a result, we have confirmed that capital utilization is too suppressed with a stricter Basel regulatory capital constraint planned to be introduced by January 1st, 2028, which corresponds to the regulatory capital credit risk, $risk^{RC}$ with conservative β in this paper. Since this upcoming constraint provides more stringent risk evaluation than the economic capital credit risk, $risk^{EC}$, the constraint on the foreign BU following a conventional way of Japanese megabanks is especially easier to be reached, and the effective use of the bank capital becomes suppressed. Consequently, applying $risk^{EC}$ to both BU and segment constraints generates non-negligible larger profits than using $risk^{RC}$. (See Table 9 and its explanation in Section 3.4.)

In addition, our result implies that if a stricter Basel regulatory capital constraint is applied to some segmented portfolio other than BUs, the capital utilization will become suppressed furthermore. Therefore, we propose to unify the internal risk constraints on the business units and industrial sectors by using credit risk amounts in terms of economic capitals.

Finally, we list up the following important topics as our future research themes: (i) As stated in Introduction section, since financial institutions, particularly international banks will be forced to construct loan portfolios in line with the promotion of de-carbonized society, we will incorporate environmental constraints into our optimization problem 2.1. (ii) From the perspective of aiming company-wide profitability, we will also consider the optimal allocation between loan and other securities such as equities. (iii) Reflecting a mid-term business plan in practice, we will extend this research to a multi-period setting. Then, even with 3% that is the currently adopted standard level of exposure change ratio, we will expect large difference in the result between before and after optimization. (iv) To examine the performance of our optimized loan portfolios, we will implement historical simulations with actual data.

AppendixA Details of industrial sectors and the exposure allocation in each BU

Let us explain the detail of exposure allocation within segments $\gamma(j,k)$. First, we consider the domestic BU ($k=1$). The domestic BU has 12 industrial sectors, i.e., $m=12$. We calculate the regulatory capital credit risk $risk^{RC}_{j,1}$ by the formula (2.14), and sort the $risk^{RC}_{j,1}$ by the industrial sector in descending order. The industrial sectors of domestic BU ($k=1$) and their $risk^{RC}_{j,1}$ are shown in Table 22, where $risk^{RC}_{1,1}$ is the largest credit risk amount within domestic BU.

Then, we set segment $\gamma(j,1)$ to be the high risk segment for $j \in (1,2,3)$, the middle risk segment for $j \in (4,5,6)$ and the low risk segment for $j \in (7,8,9,10,11,12)$. Table 23 shows the number of obligors for these three categorized segments, where the exposure size per obligor in Case 1 is greater than those in Case 2 and Case 3.

Table22 The industrial sectors of domestic BU ($k=1$) and their $risk^{RC}_{j,1}$ ($risk^{RC}_{j,k}$: 100 million yen)

j	Industrial sector of domestic BU (k=1)	$A_{j,1}$	$risk^{RC}_{j,1}$
1	Industrials	12,000	625
2	Consumer & Discretionary	8,000	487
3	Real Estate	9,000	324
4	Materials	6,000	307
5	Financials	7,000	227
6	Utilities	5,000	214
7	Health Care	3,000	163
8	Information Technology	2,500	145
9	Consumer Staples	2,500	131
10	Energy	2,000	99
11	Communication Services	1,000	59
12	Government & Other	2,000	42

Table23 Number of obligors within each segment $\gamma(j,1)$ for domestic BU

	High risk segment $\gamma(j,1)$	Middle risk segment $\gamma(j,1)$	Low risk segment $\gamma(j,1)$	domestic
Case	$j \in (1,2,3)$	$j \in (4,5,6)$	$j \in (7,8,9,10,11,12)$	Total
1	20	10	5	120
2	200	100	50	1,200
3	2,000	1,000	500	12,000

Next, we consider the foreign BU ($k=2$). The foreign BU also has 12 industrial sectors, i.e., $m=12$. We calculate the regulatory capital credit risk $risk^{RC}_{j,2}$ by the formula (2.14), and sort the $risk^{RC}_{j,2}$ by the industrial sector in descending order. The industrial sectors of the foreign BU ($k=2$) and their $risk^{RC}_{j,2}$ are shown in Table 24, where $risk^{RC}_{1,2}$ is the largest credit risk amount within foreign BU.

Then, we set the segment $\gamma(j,2)$ to be the high risk segment for $j \in (1,6,2)$, the middle risk segment for $j \in (10,8,5)$ and the low risk segment for $j \in (7,4,11,3,9,12)$. Table 25 shows the number of obligors for these three categorized segments, where the exposure size per obligor in Case 1 is greater than those in Case 2 and Case 3 as in the domestic BU.

Table24　The industrial sectors of foreign BU ($k = 2$) and their $risk^{RC}_{j,2}$　($risk^{RC}_{j,k}$: 100 million yen)

j	Industrial sector of foreign BU (k=2)	$A_{j,2}$	$risk^{RC}_{j,2}$
1	Industrials	8,000	487
6	Utilities	8,000	440
2	Consumer & Discretionary	4,000	282
10	Energy	4,000	227
8	Information Technology	3,000	206
5	Financials	4,000	169
7	Health Care	2,000	132
4	Materials	2,000	120
11	Communication Services	1,500	103
3	Real Estate	2,000	88
9	Consumer Staples	500	30
12	Government & Other	1,000	29

Table25　Number of obligors within each segment $\gamma(j, 2)$ for foreign BU

	High risk segment $\gamma(j, 2)$	Middle risk segment $\gamma(j, 2)$	Low risk segment $\gamma(j, 2)$	foreign
Case	$j \in (1, 6, 2)$	$j \in (10, 8, 5)$	$j \in (7, 4, 11, 3, 9, 12)$	Total
1	20	10	5	120
2	200	100	50	1,200
3	2,000	1,000	500	12,000

Finally, from Table 23 and Table 25, we have Table 26 below showing the number of obligors in each BU. We also note that the total number of obligors for each case 1,2,3 in Table 26 is the same as in Table 5, (e.g., a-1,a-2,a-3).

Table26　Number of obligors within each BU

Case	domestic	foreign	total
1	120	120	240
2	1,200	1,200	2,400
3	12,000	12,000	24,000

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