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Abstract

We study payoffs in the subgame perfect equilibria of general n-player discounted overlapping generations games. Without a public randomization device and the observability of mixed actions, we show that patient players can approximately obtain any payoffs in the smallest cube containing the feasible and individually rational one-shot payoffs in the sense of effective minimax values with pure actions. This result is obtained when we first choose the discount rate and then choose the players' lifespan. When mixed actions are observable, the analogous result holds true for mixed minimaxing. We also show that players cannot obtain better payoffs outside this cube defined by mixed minimaxing.

Keywords: Overlapping Generations Games, Intertemporal Trade, Discounted Games, Folk Theorem, Cubic Hull, Payoff Approximation.
JEL Classification Codes: C70, C72, C73.

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1 Introduction

Overlapping generations (OLG) games are a class of repeated games in which players in the same generation interact with each other for a sufficiently long time, and are then sequentially replaced by the successors in the next generation. The folk theorem in general n-player OLG games was first proved by Kandori [6], which stated that the players in each generation can obtain any payoffs in V^*, which is the set of feasible and individually rational one-shot payoffs. Following this study, Smith [9] proved several versions of the folk theorems with stronger results.

While it has been shown that players can obtain payoffs in V^* without discounting, and that the result is robust to low discounting in OLG games, the characterization of the equilibrium set of payoffs in the analyses of Kandori [6], Smith [9], and subsequent studies is restrictive. These results do not imply that players cannot attain payoffs outside V^*; when they discount the future, they can transfer each other's payoffs over time because the young and old generations view current payoffs differently, which allows their equilibrium payoffs to go outside V^*. Indeed, in two-player discounted OLG games, the author's recent study Morooka [8] has shown that players in each generation can obtain payoffs in the smallest rectangle which contains V^* in subgame perfect equilibria. There, we do not require players to have different discount factors, without which, as Sorin [10] showed, equilibrium payoffs never go outside V^* in the standard repeated games without overlapping generations.

We generalize this result to discounted OLG games with any n-player stage game, getting rid of the assumptions imposed in Morooka [8] that a Public Randomization Device (PRD) is available and that mixed actions are observable. In order to determine the individual rationality, the notion of effective minimaxing by Wen [11] is hired. We prove the folk theorem, stating that players in each generation can obtain payoffs in the cubic hull of V^*, the smallest n-dimensional cube which contains V^*, when they discount the future but are sufficiently patient and participate in the game for sufficiently long time. We also examine whether players can obtain better payoffs outside this cube. When players are allowed to observe mixed actions, we can expand this cube to the one defined by mixed minimax values, but it is also shown that further expansion of the equilibrium payoff set is impossible.

There are two kinds of difficulty which Morooka [8] did not have to consider but we must to obtain the present result. The first one is how to punish the deviations from minimaxing actions in the game with many players. Different from two-player case where the mutual minimaxing is available, player-specific minimax actions must be played in repeated games with three or more players, from which the punishers have an incentive to deviate. In order to avoid such deviations, the punishers must be rewarded sufficiently after the minimax phase, compared to the minimaxed player.

Another difficulty is how to get rid of a PRD and the observability of mixed actions. First, when a PRD is not available, players cannot obtain the exact value of target payoff vector by correlated one-shot actions. Instead, they must periodically play a finite sequence of pure actions which approximates the target payoff vector. Second, we restrict the players' minimax values to those by pure actions in the main theorem, owing to which players can detect the deviation from minimaxing each other.

The reasons why it is thought to be impossible to obtain the theorem as the version of mixed minimax values without observable mixed actions are as follows. It is known that Gossner [5] guaranteed the folk theorems by Kandori [6] and Smith [9] without observable mixed actions, where the minimax values are defined by mixed actions. Different from the standard repeated games where players' lifespan is infinite, we cannot use the strategy formulated by Fudenberg and Maskin [3, 4] which makes the punishers indifferent between any action by giving them a well-designed sequence of continuation payoffs lasting over the rest of the repetition after the punishing sequence. Instead, Gossner [5] developed a mathematical statistic on the history during punishing period to work out good punishers and bad ones; only the formers receive a reward at the end of their lives. However, such construction contradicts our requirement. In order to obtain our result, there is an order in the choice of parameters; we must fix the discount factor first, and then choose players' lifespan depending on the discount factor. This is because, as the discount factor approaches one, we need a longer lifespan in order for players' continuation payoffs after they become old to diminish sufficiently.

The remainder of this paper is organized as follows. In Section 2, we define the model of OLG games and prove the main theorem to obtain the cubic hull as equilibrium payoffs. In Section 3, we introduce observable mixed actions and show that the equilibrium set cannot be expanded beyond the corresponding cube. In Section 4, we discuss remaining important issues that cannot be included in this paper.

2 Main Results

2.1 Stage Games

Let $G = (N,\ A,\ g)$ be an n-player stage game defined as follows. $N = \{1, 2, \cdots, n\}$ with $n \geq 2$ is the set of players in G. For $i \in N$, A_i is i's finite set of pure actions, and $\triangle A_i$ is i's set of mixed actions. We denote i's payoff function as $g_i :\ \triangle A \to \mathbb{R}$, and let $V = co\big(g(A)\big)$ be the set of feasible payoffs, where $A = \prod_{i \in N} A_i$ and $\triangle A = \prod_{i \in N} \triangle A_i$. We define the set $N(i) \subset N$ of players whose payoff functions are equivalent to i, as follows:

$$N(i) = \{j \in N :\ \exists \phi > 0,\ \exists \psi \in \mathbb{R},\ \forall a \in A,\ g_j(a) = \phi g_i(a) + \psi\}.$$

The **pure** effective minimax actions for i is defined as follows, where $A_{-i} = \prod_{j \in N \setminus \{i\}} A_j$:

$$m^i \in \arg \max_{j \in N(i)} \min_{a_{-j} \in A_{-j}} \max_{a_j \in A_j} g_i(a_j,\ a_{-j}).$$

The set of feasible and individually rational payoffs by pure minimaxing actions is defined as $V^* = \big\{v \in V :\ \forall i \in N,\ v_i \geq g_i(m^i)\big\}$. Without loss of generality, i's minimum payoff in V^* is normalized to zero, as follows:[1]

$$\min_{v \in V^*} v_i = 0.$$

This value determines the lower bound of players' equilibrium payoffs. We also denote i's maximum payoff in V^* as r_i, defined as follows:

$$r_i = \max_{v \in V^*} v_i.$$

This value determines the upper bound of players' equilibrium payoffs in our result. We then define the **cubic hull** of V^*, say $C(V^*)$, as follows:

$$C(V^*) = \prod_{i \in N} [0,\ r_i].$$

Note that in many stage games, $C(V^*) \setminus V^* \neq \emptyset$ holds. Assuming that $V^* \cap \mathbb{R}^n_{++} \neq \emptyset$, we then define the "$\mathbb{Q}$-points" of V^* as $V^*_Q = \{v \in V^* \cap \mathbb{R}^n_{++} :\ \exists (p_a)_{a \in A} \in \triangle^{|A|} \cap \mathbb{Q}^{|A|},\ v = \sum_{a \in A} p_a g(a)\}$, where $\triangle^{|A|} = \{(p_a)_{a \in A} \in \mathbb{R}^{|A|}_+ :\ \sum_{a \in A} p_a = 1\}$ is the $|A|$-dimensional basic simplex. Finally, we define the "$\mathbb{Q}$-points" of $C(V^*)$, as follows:

$$C_Q(V^*) = \{v \in C(V^*) :\ \forall i \in N,\ \exists v^i_{-i} \in \mathbb{R}^{n-1},\ (v_i,\ v^i_{-i}) \in V^*_Q\}.$$

Before proceeding to the next subsection, we observe the following lemma, which is derived directly by Lemma 2 along with the proof of Theorem 1 of Abreu et al. [1].

Lemma 1. Suppose $V^* \cap \mathbb{R}^n_{++} \neq \emptyset$. Then $V^*_Q \neq \emptyset$ and for any payoffs $v \in V^*_Q$, there exists $K \in \mathbb{N}$ and sequences of pure actions $(a(t))^K_{t=1}$ and $(c^i(t))^K_{t=1}$ for $i \in N$ which satisfy the following equations:

$$Kv = \sum_{t=1}^{K} g(a(t))\ (target\ payoff\ generation\ by\ finite\ sequence\ of\ pure\ actions),$$

[1] This implies that for some $\phi > 0$, $g_j(a) = \phi g_i(a)$ holds for all $a \in A$ if $j \in N(i)$, because $0 = g_j(m^j) = g_j(m^i) = \phi g_i(m^i) + \psi = \psi$ must be satisfied.

$$\sum_{t=1}^{K} g(c^i(t)) >> \mathbf{0} \ \textit{for all } i \in N \ (\textit{strict individual rationality}),$$

$$\sum_{t=1}^{K} g_i(c^i(t)) < Kv_i \ \textit{for all } i \in N \ (\textit{target payoff domination}),$$

$$\sum_{t=1}^{K} g_i(c^i(t)) < \sum_{t=1}^{K} g_i(c^j(t)) \ \textit{for all } i \in N \textit{ and } j \in N \setminus N(i) \ (\textit{payoff asymmetry}).$$

2.2 OLG Games

Given a stage game G defined above, we construct the OLG game with perfect monitoring, $OLG(G;\ \delta,\ T)$ as follows (also see Table 1):

·The game starts in period 1. In every period, G is played by n finitely-lived players.

·For $i \in N$ and $d \geq 1$, the player with A_i in generation d joins in the game at the beginning of period $(d-1)\overline{T}_n + \overline{T}_{i-1} + 1$, and lives for the following $\overline{T}_n$ periods, until he retires at the end of period $d\overline{T}_n + \overline{T}_{i-1}$, where $T = (T_1, T_2, \cdots, T_n) \in \mathbb{N}^n$, $\overline{T}_0 = 0$ and $\overline{T}_i = \sum_{j=1}^{i} T_j$ for $i \in N$. The only exceptions are the players with A_i for $i \in N \setminus \{1\}$ in generation 0, who participates in the game between periods 1 and $\overline{T}_{i-1}$.

·Each player's per-period payoffs are discounted at a common $\delta \in (0,\ 1]$.[2]

<table>
<tr><td>Period</td><td>$1 \sim \overline{T}_1$</td><td>$\overline{T}_1+1 \sim \overline{T}_2$</td><td>$\overline{T}_2+1 \sim \overline{T}_3$</td><td>$\overline{T}_3+1 \sim \overline{T}_3+\overline{T}_1$</td><td>$\overline{T}_3+\overline{T}_1+1 \sim \overline{T}_3+\overline{T}_2$</td><td>$\overline{T}_3+\overline{T}_2+1 \sim 2\overline{T}_3$</td><td>$2\overline{T}_3+1 \sim 2\overline{T}_3+\overline{T}_1$</td><td>$\cdots$</td></tr>
<tr><td>A_1</td><td colspan="3">Generation 1</td><td colspan="3">Generation 2</td><td colspan="2">Generation 3 $\cdots$</td></tr>
<tr><td>A_2</td><td>Generation 0</td><td colspan="3">Generation 1</td><td colspan="3">Generation 2</td><td>$\cdots$</td></tr>
<tr><td>A_3</td><td colspan="2">Generation 0</td><td colspan="3">Generation 1</td><td colspan="3">Generation 2 $\cdots$</td></tr>
</table>

Table 1: Structure of OLG game with $n = 3$

In every period, players can only observe each other's **realized pure actions**. When a sequence of actions $\big(\alpha(t)\big)_{t=1}^{\overline{T}_n} \in (\triangle A)^{\overline{T}_n}$ is played throughout a player's life with A_i, his average payoff is as follows:[3]

$$\frac{1}{\sum_{t=1}^{\overline{T}_n} \delta^{t-1}} \sum_{t=1}^{\overline{T}_n} \delta^{t-1} g_i\big(\alpha(t)\big).$$

2.3 The Folk Theorem

We first prove the following result.

Lemma 2. (Approximation of payoffs in $C_Q(V^*)$)
Suppose $V^* \cap \mathbb{R}^n_{++} \neq \emptyset$. For every payoffs $v \in C_Q(V^*)$, and for every $\epsilon > 0$, there exists $\underline{\delta} \in (0,\ 1)$ such that if $\delta \in [\underline{\delta},\ 1)$, then for every sufficiently large $T \in \mathbb{N}^n$ dependent on δ, there exists a subgame perfect equilibrium in $OLG(G;\ \delta,\ T)$ where the players in each generation $d \geq 1$ obtain average payoffs w that satisfies $|w_i - v_i| < \epsilon$ for $i \in N$.

Proof. See the appendix.

[2]Although our result does not hold with $\delta = 1$, it is convenient to consider this case in order to construct the strict punishments under no discount that are still available with $\delta < 1$.

[3]For the player with A_i for $i \in N \setminus \{1\}$ in generation 0, replace $\overline{T}_n$ with $\overline{T}_{i-1}$.

It is then straightforward to get the following theorem.

Theorem 1. (A folk theorem in n-player discounted OLG games with pure minimax values) Suppose $V^* \cap \mathbb{R}^n_{++} \neq \emptyset$. For every payoffs $v \in C(V^*)$, and for every $\epsilon > 0$, there exists $\underline{\delta} \in (0,\ 1)$ such that if $\delta \in [\underline{\delta},\ 1)$, then for every sufficiently large $T \in \mathbb{N}^n$ dependent on δ, there exists a subgame perfect equilibrium in $OLG(G;\ \delta,\ T)$ where the players in each generation $d \geq 1$ obtain average payoffs w that satisfies $|w_i - v_i| < \epsilon$ for $i \in N$.

Proof. Choose any $v \in C(V^*)$ and $\epsilon > 0$. Because $closure(C_Q(V^*)) = C(V^*)$, there exists $v' \in C_Q(V^*)$ satisfying that $|v'_i - v_i| < 0.5\epsilon$ for $i \in N$. Lemma 2 then guarantees that for $\delta \in [\underline{\delta},\ 1)$ for some $\underline{\delta} \in (0,\ 1)$ and sufficiently large $T \in \mathbb{N}^n$, there exists a subgame perfect equilibrium payoff vector w in $OLG(G;\ \delta,\ T)$ with $|w_i - v'_i| < 0.5\epsilon$. Therefore we get $|w_i - v_i| \leq |w_i - v'_i| + |v'_i - v_i| < \epsilon$, completing the proof.□

3 Observable Mixed Actions and the Impossibility of Further Extension

3.1 Mixed Minimax Values

So far, we have seen that players can obtain equilibrium payoffs in $C(V^*)$ in discounted OLG games. Here, some readers may wonder whether they can obtain better payoffs outside $C(V^*)$ in equilibria. We first show that the set of equilibrium payoffs can be expanded when mixed actions are observable. Here, the **mixed** effective minimax actions for $i \in N$ is defined as follows, where $\triangle A_{-i} = \prod_{j \in N \setminus \{i\}} \triangle A_j$:

$$\mu^i \in \arg \max_{j \in N(i)} \min_{\alpha_{-j} \in \triangle A_{-j}} \max_{a_j \in A_j} g_i(a_j,\ \alpha_{-j}).$$

The mixed version of the set of feasible and individually rational payoffs is defined as $V^{**} = \{v \in V :\ \forall i \in N,\ v_i \geq g_i(\mu^i)\}$. Assuming that $\min_{v \in V^{**}} v_i = 0$ for $i \in N$ and $V^{**} \cap \mathbb{R}^n_{++} \neq \emptyset$, the cubic hull of V^{**} is defined as $C(V^{**}) = \prod_{i \in N} [0,\ \rho_i]$, where $\rho_i = \max_{v \in V^{**}} v_i$. Note that (a translation with respect to the origin of) $C(V^*)$ is included in this set. It is straightforward to derive the analogue of Theorem 1 for $C(V^{**})$.

Lemma 3. (Equilibrium payoffs in n-player discounted OLG games with observable mixed actions) Suppose $V^* \cap \mathbb{R}^n_{++} \neq \emptyset$ and that mixed actions are observable in every period. Then, for every payoffs $v \in C(V^{**})$, and for every $\epsilon > 0$, there exists $\underline{\delta} \in (0,\ 1)$ such that if $\delta \in [\underline{\delta},\ 1)$, then for every sufficiently large $T \in \mathbb{N}^n$ dependent on δ, there exists a subgame perfect equilibrium in $OLG(G;\ \delta,\ T)$ where the players in each generation $d \geq 1$ obtain average payoffs w that satisfies $|w_i - v_i| < \epsilon$ for $i \in N$.

Proof. Replacing V^* with V^{**} and m^i with μ^i for $i \in N$ in the proof of Theorem 1 gives the exact proof of this lemma. Now Steps 5 and 6 need the observability of mixed actions in order to detect the deviation from minimaxing, which can be a mixed action.□

3.2 Impossibility of a Further Extension

Finally, we show that the set of equilibrium payoffs cannot be further extended from $C(V^{**})$, independent of δ, T, and the observability of mixed actions. Here, an equilibrium[4] in $OLG(G;\ \delta,\ T)$ is called **periodic** if a sequence of actions $(\alpha(t))_{t=1}^{\overline{T}_n} \in \triangle A^{\overline{T}_n}$ is played on the path periodically, between period $(d-1)\overline{T}_n + 1$ and $d\overline{T}_n$ for each $d \geq 1$.

Theorem 3. (Impossibility of Further Extension outside $C(V^{**})$) For every $\delta \in (0,\ 1]$, for every $T \in \mathbb{N}^n$, and for every periodic equilibrium in $OLG(G;\ \delta,\ T)$, its

[4]We do not require that the equilibrium is subgame perfect in this result.

average payoff vector v of the players in each generation $d \geq 1$ satisfies $v \in C(V^{**})$. Especially when $\delta = 1$, $v \in V^{**}$ holds.

Proof. Because the equilibrium is periodic, the total payoff vector obtained throughout the periods when a player with the action set A_i is the youngest can be written as x^i (see also Table 2).[5]

Duration	T_1	T_2	T_3	T_1	T_2	T_3	T_1	$\cdots$
A_1	x_1^1	x_1^2	x_1^3	x_1^1	x_1^2	x_1^3	x_1^1	$\cdots$
A_2	x_2^1	x_2^2	x_2^3	x_2^1	x_2^2	x_2^3	x_2^1	$\cdots$
A_3	x_3^1	x_3^2	x_3^3	x_3^1	x_3^2	x_3^3	x_3^1	$\cdots$

Table 2: Sequence of total payoffs with $n = 3$

By the optimality of equilibria, the continuation payoffs must satisfy the following individual rationality for all $i \in N$, where $f(0) = 0$, $f(t) = t$ for $t \leq n$, $f(t) = t - n$ for $t \geq n + 1$ and $T_0 = 0$:

$$\sum_{t=0}^{n-1} \delta^{\sum_{s=0}^{t} T_{f(i+s-1)}} x_i^{f(i+t)} \geq 0,$$

$$\sum_{t=0}^{n-1} \delta^{\sum_{s=0}^{t} T_{f(i+s-1)}} x_i^{f(i+t)} \geq \sum_{t=0}^{k-1} \delta^{\sum_{s=0}^{t} T_{f(i+s-1)}} x_i^{f(i+t)} \quad for\ all\ k \in \{1, 2, \cdots, n-1\}.$$

Therefore, for all $i \in N$ and $j \in N \setminus N(i)$, the following relationship must hold:[6]

$$\sum_{t=0}^{n-1} \delta^{\sum_{s=0}^{t} T_{f(i+s-1)}} x_j^{f(i+t)} \geq 0.$$

This implies that during the life of each player with A_i, payoffs of all his opponents (whose payoff functions are not equivalent to g_i) are individually rational, which in turn guarantees that his total payoff cannot exceed $\sum_{t=1}^{\overline{T}_n} \delta^{t-1} \rho_i$ in equilibria. Finally, when $\delta = 1$, the average payoff vector satisfies the following inequality, completing the proof:

$$\mathbf{0} \leq \frac{1}{\overline{T}_n} \sum_{i=1}^{n} x^i \in V^{**}.$$

□

4 Discussion and Conclusion

In this study, we analyzed a model of an n-player discounted OLG game. We observed that players can obtain payoffs outside V^*. Some readers may believe that it will be convenient to compute the exact shape of the equilibrium payoff set of OLG games without a PRD for arbitrarily fixed δ and T. For standard repeated games, Lehrer and Pauzner [7] provided an algorithm to find the shape of the frontier using a PRD as players become patient. Dasgupta and Ghosh [2] provide a condition to identify all payoff profiles that can be obtained as subgame perfect equilibrium payoffs of some discount factor profile without using a PRD. When we apply these results to OLG games, however, recursive methods which can be used in the standard repeated games will be no longer available. We leave these issues to be resolved in future research.

[5] For example, when $n = 2$, $T_1 = 2$ and players obtain one-shot payoffs $(1,\ 2)$ in period 1 and $(3,\ 4)$ in period 2, $x^1 = (1 + 3\delta,\ 2 + 4\delta)$.

[6] When $n = 3$, for example, $x_2^1 \geq 0$ and $x_2^2 + \delta^{T_2} x_2^3 + \delta^{T_2+T_3} x_2^1 \geq 0$ must be satisfied, which yields $x_2^1 + \delta^{T_1} x_2^2 + \delta^{T_1+T_2} x_2^3 = (1 - \delta^{T_1+T_2+T_3}) x_2^1 + \delta^{T_1} (x_2^2 + \delta^{T_2} x_2^3 + \delta^{T_2+T_3} x_2^1) \geq 0$. Analogously, $x_3^1 + \delta^{T_1} x_3^2 + \delta^{T_1+T_2} x_3^3 = (1 - \delta^{T_1+T_2+T_3})(x_3^1 + \delta^{T_1} x_3^2) + \delta^{T_1+T_2} (x_3^3 + \delta^{T_3} \delta_3^1 + \delta^{T_3+T_1} x_3^2) \geq 0$ must be satisfied. These imply that the total payoff of players with A_1 cannot exceed $\sum_{t=1}^{T_1+T_2+T_3} \delta^{t-1} \rho_1$.

Appendix: Proof of Lemma 2

Before the formal proof, we use the following Table 3 and briefly see how the players in each generation can obtain good equilibrium payoffs $(v_1^1, v_2^2, \cdots, v_n^n) \in C(V^*)$, which are possibly outside V^*. For $i \in N$, let $b^i \in \arg\max_{a \in A} g_i(a)$ be the actions yielding the best one-shot payoff to the player with A_i, and we denote a one-shot Nash profile as e.

Duration	$\ell_1 K$	S	$T_1 - \ell_1 K - S$	$\ell_2 K$	S	$T_2 - \ell_2 K - S$	$\ell_3 K$	S	$T_3 - \ell_3 K - S$	$\ell_1 K$	S	$T_1 - \ell_1 K - S$	$\cdots$
A_1	$\boldsymbol{v_1^1}$	$g_1(b^2)$	$g_1(e)$	v_1^2	$g_1(b^3)$	$g_1(e)$	v_1^3	$g_1(b^1)$	$g_1(e)$	$\boldsymbol{v_1^1}$	$g_1(b^2)$	$g_1(e)$	$\cdots$
A_2	v_2^1	$g_2(b^2)$	$g_2(e)$	$\boldsymbol{v_2^2}$	$g_2(b^3)$	$g_2(e)$	v_2^3	$g_2(b^1)$	$g_2(e)$	v_2^1	$g_2(b^2)$	$g_2(e)$	$\cdots$
A_3	v_3^1	$g_3(b^2)$	$g_3(e)$	v_3^2	$g_3(b^3)$	$g_3(e)$	$\boldsymbol{v_3^3}$	$g_3(b^1)$	$g_3(e)$	v_3^1	$g_3(b^2)$	$g_3(e)$	$\cdots$

Table 3: Equilibrium payoff path with $n = 3$

As you can see in this table, each player with A_i obtains the target payoff v_i^i in his first $\ell_i K$ periods of life. Here, his opponents endure relatively low payoffs v_{-i}^i so that players' payoffs during these periods are feasible and strictly individually rational, i.e., $(v_i^i,\ v_{-i}^i) \in V^* \cap \mathbb{R}_{++}^n$. In the following S periods, he gives the maximum one-shot payoff $g_j(b^j)$ to the oldest opponent, in order to block the deviation by the opponent who is going to retire. After his opponent is replaced by a new player, he helps the new youngest opponent to obtain v_j^j for $\ell_j K$ periods. After becoming the oldest, he receives a terminal bonus $g_i(b^i)$ for S periods before the retirement. When he discounts the future and his lifespan is sufficiently long, his continuation payoff after he becomes old vanishes, and the average payoff throughout his life becomes almost equal to what he obtains in early $\ell_i K$ periods of his life, v_i^i.

Now we proceed to formally prove Lemma 2.

Proof. Choose any $v = (v_1^1, v_2^2, \cdots, v_n^n) \in C_Q(V^*)$ and $\epsilon > 0$. By the definition of $C_Q(V^*)$, there exists v_{-i}^i which satisfies $v^i = (v_i^i,\ v_{-i}^i) \in V_Q^*$ for $i \in N$. By Lemma 1, we can find $K \in \mathbb{N}$ and sequences of pure actions $(a^i(t))_{t=1}^K$ and $(c^{i,j}(t))_{t=1}^K$ for $i,\ j \in N$ which satisfy the following equations:[7]

$$Kv^i = \sum_{t=1}^{K} g(a^i(t)) \ for\ i \in N,$$

$$\sum_{t=1}^{K} g(c^{i,j}(t)) >> \mathbf{0} \ for\ i,\ j \in N,$$

$$\sum_{t=1}^{K} g_j(c^{i,j}(t)) < Kv_j^i \ for\ i,\ j \in N,$$

$$\sum_{t=1}^{K} g_j(c^{i,j}(t)) < \sum_{t=1}^{K} g_j(c^{i,k}(t)) \ for\ i,\ j \in N\ and\ k \in N \setminus N(j).$$

When the player with A_j deviates right before the retirement of his opponent with A_i, players play the sequence $c^{i,j}$ instead of a^i to punish him after the player with A_i is replaced.

The game starts in period 1 with playing $a^1(1)$. The natural numbers q_1, q_2, S, and q_3 are determined later in this order so that the following strategy profile is well-defined and subgame perfect. Here, we consider the strategy profile for any T_i-period block between $(d-1)\overline{T}_n + \overline{T}_{i-1} + 1$ and $(d-1)\overline{T}_n + \overline{T}_i$ for $i \in N$ and $d \geq 1$, and period $(d-1)\overline{T}_n + \overline{T}_{i-1} + t$ is just called "period t" for $t \in \{1, 2, \cdots, T_i\}$, during which the player with A_i is the youngest. The profile consists of 7 steps. We assume that the action set of the oldest player in the current block is A_j.

[7] When we omit i and replace j (resp. k) with i (resp. j), we get exactly the same equations as in Lemma 1.

First, we define the following main-path strategy with $n+1$ states $\lambda \in \{Good\} \cup \{Late(1), Late(2), \cdots, Late(n)\}$. In *Good* state, the youngest player obtains the target payoff, whereas $Late(k)$ state is provided to punish the late deviation of the player with A_k for $k \in N$. The initial state is $\lambda =$ "$Good$".

·Step 1. In period $t \leq q_3 K$, players play $a^i(h(t))$ in state $\lambda =$ "$Good$", and play $c^{i,k}(h(t))$ in state $\lambda =$ "$Late(k)$", where $h(sK + \tau) = \tau$ for $\tau \in \{1, 2, \cdots, K\}$ and $s \in \mathbb{N} \cup \{0\}$. At the end of period $q_3 K$, the state is reset to $\lambda =$ "$Good$".
·Step 2. In period $t \in \{q_3 K + 1, \cdots, \ell_i K\}$, players play $a^i(h(t))$.
·Step 3. In period $t \in \{\ell_i K + 1, \cdots, \ell_i K + S\}$, players play b^j.
·Step 4. In period $t \geq \ell_i K + S + 1$, players play e.

After someone unilaterally deviates, players play the following punishments. The punishments are also applied to the deviations from themselves. After each punishment, players return to main-path.

·Step 5. When a player with A_k deviates in period $t \leq \ell_i K - (q_1 + q_2)K$, players play e for $K - h(t)$ periods, until the end of period τ with $h(\tau) = K$. Players then play m^k for the following $q_1 K$ periods. After that, during the following $q_2 K$ periods, they play $c^{i,k}(s)$ in period s.
·Step 6. When the oldest player deviates in period $t \in \{\ell_i K - (q_1 + q_2)K + 1, \cdots, \ell_i K\}$, players play m^j until the end of period $\ell_i K + S$.
·Step 7. When a young player with A_k deviates in period $t \in \{\ell_i K - (q_1 + q_2)K + 1, \cdots, \ell_i K + S\}$, players play e until the oldest player retires, and the state in the next block is set to $\lambda =$ "$Late(k)$".

According to this profile, the K-period sequence a^i is played ℓ_i times on the equilibrium path when each player with A_i is the youngest, and his average payoff is almost equal to v_i^i under the positive rate of discounting $\delta < 1$ and sufficiently large $\ell_i K$ relative to S, as you can see in Table 3.

In the remainder of this section, it is proved that the above strategy profile forms a subgame perfect equilibrium for δ sufficiently close to 1 when we choose q_1, q_2, S, and q_3 appropriately. The methodology is to provide strictly positive penalties for any unilateral deviation from each step when $\delta = 1$. Then, by continuity of payoffs, there is some $\underline{\delta} \in (0,\ 1)$ for which all penalties are still positive and Steps 1 and 2 still approximate the target payoff[8] for $\delta \in [\underline{\delta},\ 1)$. In determining the parameters, we consider the "worst-case scenario," where the incentive to deviate is greatest.

·Deviations in period $t \leq \ell_i K - (q_1 + q_2)K$ when nobody is minimaxed:
Let $\underline{u} = \min_{i,j \in N} \{v_j^i,\ \sum_{t=1}^{K} g_j(c^{i,j}(t))/K\} > 0$ be the worst per-period payoff in Steps 1 and 2. When a player deviates, he gets at most $K\beta$ immediately and the following $K - 1$ periods, where $\beta = \max_{i \in N} \{g_i(b^i)\}$ is the maximum payoff for players. However, his payoff during the next $q_1 K$ periods is reduced to 0 by Step 5. When he does not deviate, on the other hand, he can get at least $K\omega + q_1 \underline{u}$ during the same periods, where $\omega = \min_{i \in N, a \in A} \{g_i(a)\}$ is the minimum payoff for players. In order to block the deviation, we choose q_1 to satisfy the following inequality:

$$K\omega + q_1 \underline{u} > K\beta.$$

·Deviations from minimaxing of Step 5 in period $t \leq \ell_i K - (q_1 + q_2)K$:
When a minimaxing player with A_k deviates from m^j satisfying $k \in N \setminus N(j)$,[9] he gets at most $K\beta$ immediately and the following $K - 1$ periods, and gets 0 during the next $q_1 K$ periods. After that, he gets $q_2 \sum_{t=1}^{K} g_k(c^{i,k}(t))$ in the following $q_2 K$ periods. When he does not deviate, on the other hand, he can get at least $K\omega + q_1\omega + q_2 \sum_{t=1}^{K} g_k(c^{i,j}(t))$ during the same periods. In order to block the deviation, we choose q_2 to satisfy the following inequality for $i,\ k \in N$:

$$K\omega + q_1\omega + q_2 \sum_{t=1}^{K} g_k(c^{i,j}(t)) > K\beta + q_2 \sum_{t=1}^{K} g_k(c^{i,k}(t)).$$

[8] When a PRD is available, we need not concern about this issue of approximation, because the PRD can generate exactly the same value as the target payoff every period.

[9] When $k \in N(j)$, he has no incentive to deviate because m_k^j is the best response to m_{-k}^j.

·Deviations by the oldest player in period $t \geq \ell_i K - (q_1 + q_2)K + 1$:
Suppose that the oldest player deviates in period $t = \ell_i K - (q_1 + q_2)K + 1$. He gets at most β immediately, but his continuation payoff until the end of period $\ell_i K + S$ is at most 0 because of Step 6. When he does not deviate, his payoff during these $(q_1 + q_2)K + S$ periods is at least $(q_1 + q_2)K\omega + S\underline{\beta}$, where $\underline{\beta} = \min_{i \in N}\{g_i(b^i)\}$. We can avoid the deviation when we choose S satisfying the following inequality:

$$(q_1 + q_2)K\omega + S\underline{\beta} > \beta.$$

·Deviations by a young player in period $t \geq \ell_i K - (q_1 + q_2)K + 1$:
Suppose that a young player with A_k deviates in period $t = \ell_i K - (q_1 + q_2)K + 1$. He gets at most $((q_1 + q_2)K + S)\beta$ immediately and the following $(q_1 + q_2)K + S - 1$ periods. After the oldest player's retirement, his payoff in the following $q_3 K$ periods is reduced to at most $q_3 \sum_{t=1}^{K} g_k(c^{j,k}(t))$ because of Step 7 and Step 1 with state $\lambda =$ "$Late(k)$". When he does not deviate, he gets at least $((q_1 + q_2)K + S)\omega$ until the end of period $\ell_i K + S$, and at least $q_3 K v_k^j$ after the oldest player's retirement with state $\lambda =$ "$Good$". We can avoid the deviation when we choose q_3 satisfying the following inequality for $j,\ k \in N$:

$$((q_1 + q_2)K + S)\omega + q_3 K v_k^j > ((q_1 + q_2)K + S)\beta + q_3 \sum_{t=1}^{K} g_k(c^{j,k}(t)).$$

Therefore, deviations from any steps strictly decrease deviators' payoffs under no discount. It means that there exists some $\underline{\delta} \in (0,\ 1)$, the strategy profile described above is still subgame-perfect and satisfies the following inequalities for all $i \in N$:

$$\frac{1 - \underline{\delta}^K}{1 - \underline{\delta}} v_i^i - \frac{\epsilon}{3} < \sum_{t=1}^{K} \underline{\delta}^{t-1} g_i(a^i(t)) < \frac{1 - \underline{\delta}^K}{1 - \underline{\delta}} v_i^i + \frac{\epsilon}{3}.$$

Fix any $\delta \in [\underline{\delta},\ 1)$. Choose any $\underline{\ell} \in \mathbb{N}$ which satisfies the following inequalities:

$$\underline{\ell} > \max\{q_1 + q_2,\ q_3\},$$

$$\frac{\delta^{\underline{\ell}K}}{1 - \delta} \max\{\beta,\ -\omega\} < \frac{\epsilon}{3}.$$

We then choose a vector $T >> (\underline{\ell}K + S,\ \underline{\ell}K + S,\ \cdots,\ \underline{\ell}K + S)$. Finally, we choose $(\ell_i)_{i \in N}$ to satisfy $0 \leq T_i - \ell_i K - S \leq K - 1$ for each $i \in N$ so that Step 4 lasts for at most $K - 1$ periods. In $OLG(G;\ \delta,\ T)$, the above strategy profile is well-defined and the average payoff of players with A_i except generation 0 is at least as follows:

$$\frac{1 - \delta}{1 - \delta^{\overline{T}_n}} \{\sum_{s=1}^{\ell_i} \delta^{(s-1)K} \sum_{t=1}^{K} \delta^{t-1} g_i(a^i(t)) + \frac{\delta^{\ell_i K}}{1 - \delta}\omega\}$$

$$> \frac{1}{1 - \delta^{\overline{T}_n}} \{(1 - \delta) \sum_{s=1}^{\ell_i} \delta^{(s-1)K} (\frac{1 - \delta^K}{1 - \delta} v_i^i - \frac{\epsilon}{3}) + \delta^{\ell_i K}\omega\}$$

$$= v_i^i - \frac{\delta^{\ell_i K}(1 - \delta^{\overline{T}_n - \ell_i K})}{1 - \delta^{\overline{T}_n}} v_i^i - \frac{\sum_{t=1}^{\ell_i} \delta^{(t-1)K}}{\sum_{t=1}^{\overline{T}_n} \delta^{t-1}} \frac{\epsilon}{3} + \frac{\delta^{\ell_i K}}{1 - \delta^{\overline{T}_n}}\omega$$

$$> v_i^i - \frac{\delta^{\underline{\ell}K}}{1 - \delta}\beta - \frac{\epsilon}{3} + \frac{\delta^{\underline{\ell}K}}{1 - \delta}\omega$$

$$> v_i^i - \frac{\epsilon}{3} - \frac{\epsilon}{3} - \frac{\epsilon}{3} = v_i^i - \epsilon.$$

Analogously, their maximum average payoff is less than $v_i^i + \epsilon$, completing the proof of the lemma.□

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