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Abstract

We investigate the problem of equilibrium price formation in an incomplete securities market. Each financial firm (agent) tries to minimize its cost via continuous-time trading with a securities exchange while facing the systemic and idiosyncratic noises as well as the stochastic order-flows from its over-the-counter clients. We have shown, in the accompanying paper (Fujii & Takahashi) [19], that the solution to a certain forward backward stochastic differential equation of conditional McKean-Vlasov type gives a good approximate of the equilibrium price which clears the market in the large population limit. In this work, we prove the existence of a unique market clearing equilibrium among the heterogeneous agents of finite population size. We show the strong convergence to the corresponding mean-field limit given in [19] under suitable conditions. In particular, we provide the stability relation between the market clearing price for the heterogeneous agents and that for the homogeneous mean-field limit.

Keywords : equilibrium in incomplete markets, common noise, market clearing, price formation, mean field games

1 Introduction

The problem of equilibrium asset price formation has been one of the central issues in financial economics. For complete markets, significant advances have been made and there exists large amount of literature. See, for example, Chapter 4 of Karatzas & Shreve [27] and references therein. We also mention Kramkov [29] as a recent development.

On the other hand, the problem of equilibrium price formation in incomplete markets with continuous-time stochastic setting is still under active research. It is mainly because the traditional approach of constructing a representative agent turns out to be much more difficult than in the complete case. See Part 2 and 3 of Jarrow [26] for a nice review of the issues. Let us refer to the works [7, 6, 8, 25, 35, 42, 43, 45] which tackle the equilibrium problem in incomplete markets. In the majority of works, each agent is supposed to maximize the exponential utility function with respect to the terminal wealth. When a simple form of asset price process is assumed, it is well known that the individual optimization problem for the exponential utility function gives rise to a quadratic-growth backward stochastic differential equation (qg-BSDE).

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Under the diffusion setup, the existence of a unique solution for a qg-BSDE was solved by Kobylanski [30]. In the presence of jumps, a quadratic-exponential growth BSDE with random Poisson measures needs to be solved [40]. See also [18, 28] for recent developments. Thanks to these results, there is no problem in solving the optimization problem for an individual agent. However, when we impose the market clearing condition in the presence of multiple agents, it ends up with a coupled system of multi-dimensional qg-BSDEs. A coupled system of qg-BSDEs has long been unsolved since the traditional approach of [30] crucially relies on the comparison theorem which is, in general, unavailable for a multi-dimensional setup. In this regard, we refer to Xing & Žitković [42] as an important progress toward the solution for this problem. Some interesting applications of the result to the problem of market equilibrium can be found in the references given above.

Recent developments of Mean Field Game (MFG) theory have opened a new interesting approach to multi-agent problems. Since the publication of the pioneering works by Lasry & Lions [36, 37, 38] and Huang, Malhame & Caines [24], mean field game theory has been one of the most active research topics in various fields. The strength of the mean field approach resides in the fact that, in the large population limit, it decomposes notoriously difficult problem of a stochastic differential game into a tractable individual optimization problem and an additional fixed-point problem. It has been proved that the solution to the mean-field game equilibrium gives an ϵ-Nash equilibrium for the corresponding game of finite homogeneous agents. For interested readers, there are excellent monographs such as [2, 20, 21, 31] for analytic approach and [4, 5] for probabilistic approach. See also [3, 10, 11, 32, 33, 34] for another approach using the concept of relaxed controls, which does not produce any equation characterizing the equilibrium solution but can significantly weaken the regularity assumptions we need. Since the mean-field game theory has been constructed for the analysis of the Nash equilibrium, examples of its direct applications to the market clearing equilibrium are very hard to find. In the majority of works, certain phenomenological approaches are taken. One popular approach is to assume that the asset price process is decomposed into two parts, one is an exogenous process which is independent of the agents' action, and the other representing the market friction (i.e. price impact) which is often proportional to the average trading speed of the agents. Although this assumption makes the setup nicely fit to the Nash game, the market clearing equilibrium cannot be investigated anymore. Another approach is to impose the market clearing condition but the demand of the asset is assumed to be given by an exogenous function of price without considering the optimization problem among the agents. See [1, 9, 14, 15, 16, 23, 39, 12, 13] as interesting applications to, optimal trading, optimal liquidation, optimal oil production, and price formation in electricity markets etc., using the phenomenological approaches explained above. A notable exception directly dealing with the market clearing equilibrium is [22], where the market price process becomes deterministic due to the absence of the common noise.

In this paper, we investigate the problem of equilibrium price formation in an incomplete securities market. Each financial firm (agent) tries to minimize its cost via continuous-time trading with a securities exchange while facing the systemic and idiosyncratic noises as well as the stochastic order-flows from its over-the-counter clients. The biggest difference from the existing works is the generality of the assumption on the market price process. The price process of the n securities $(\varpi_t)_{t\in[0,T]}$ is only required to be a square integrable and progressively measurable process with respect to the full filtration. This is in clear contrast to the vast majority of works, where the asset price process is supposed to have a simplistic diffusion form. There, the volatility is often assumed to be constant and then the analysis is essentially restricted to the risk-premium term. The cost function we adopt is a natural generalization of those used in optimal liquidation problems. The running as well as the terminal costs depend not only on the storage level but also on the equilibrium price ϖ which is to be determined endogenously. The characterizing equation is a coupled system of fully-coupled forward backward stochastic differential equations (FBSDEs) instead of the coupled qg-BSDEs. The existence of unique solution is proved by exploiting the convexity as well as monotone conditions of the coefficient functions. We have shown, in the accompanying paper (Fujii & Takahashi) [19], that the solution to a certain FBSDE of conditional McKean-Vlasov type gives a good approximate of the equilibrium price. In fact, it is shown to clear the market asymptotically in the large population limit. In the current paper, we build a direct bridge connecting the game among the finite number of agents and its large population limit. We show the strong convergence to the mean-field limit given in [19] under suitable conditions. In particular, we provide the stability relation between the market clearing price for the heterogeneous agents and that for the homogeneous mean-field limit. The stability result tells us how far the equilibrium price among the heterogeneous agents deviates from the homogeneous mean-field limit. Finally, the convergence to the mean-field limit reveals a role of the securities market as an efficient filter which removes the idiosyncratic noises from the equilibrium price. This implies that each agent needs only the common market information as well as its own idiosyncratic information to implement the optimal strategy without any access to the idiosyncratic information of the others. Note that we cannot observe this feature in the setups with finite number of agents since we cannot restrict the filtration to which the price process is adapted to the one generated by the common noise. The unnatural and awkward assumption of the perfect information among the agents is thus resolved in the limit of large population.

The organization of the paper is as follows: In Section 2, the notations used in the paper are explained. In Section 3, the first major result regarding the existence of the unique equilibrium among the finite number of agents is given (Theorems 3.2 and 3.3). Section 4 is devoted to prove the strong convergence of the finite-agent equilibrium to its mean field limit (Theorem 4.2), which is the second major result of the paper. The stability result between the equilibrium price for the finite heterogeneous agents and the mean field limit of homogeneous agents is also given (Theorem 4.3). We conclude in Section 5 with some discussions on the possible directions for the future research.

2 Notation

We use the same notations adopted in the work [19]. We introduce (N+1) complete probability spaces:

$$(\overline{\Omega}^0, \overline{\mathcal{F}}^0, \overline{\mathbb{P}}^0) \quad \text{and} \quad (\overline{\Omega}^i, \overline{\mathcal{F}}^i, \overline{\mathbb{P}}^i)_{i=1}^N ,$$

endowed with filtrations $\overline{\mathbb{F}}^i := (\overline{\mathcal{F}}^i_t)_{t\geq 0}$, $i \in \{0, \cdots, N\}$. Here, $\overline{\mathbb{F}}^0$ is the completion of the filtration generated by d^0-dimensional Brownian motion $\boldsymbol{W}^0$ (hence right-continuous) and, for each $i \in \{1, \cdots, N\}$, $\overline{\mathbb{F}}^i$ is the complete and right-continuous augmentation of the filtration generated by d-dimensional Brownian motions $\boldsymbol{W}^i$ as well as a $\boldsymbol{W}^i$-independent n-dimensional square-integrable random variables (ξ^i). We also introduce the product probability spaces

$$\Omega^i = \overline{\Omega}^0 \times \overline{\Omega}^i, \quad \mathcal{F}^i, \quad \mathbb{F}^i = (\mathcal{F}^i_t)_{t\geq 0}, \quad \mathbb{P}^i \ , i \in \{1, \cdots, N\}$$

where $(\mathcal{F}^i, \mathbb{P}^i)$ is the completion of $(\overline{\mathcal{F}}^0 \otimes \overline{\mathcal{F}}^i, \overline{\mathbb{P}}^0 \otimes \overline{\mathbb{P}}^i)$ and $\mathbb{F}^i$ is the complete and right-continuous augmentation of $(\overline{\mathcal{F}}_t^0 \otimes \overline{\mathcal{F}}_t^i)_{t\geq 0}$. In the same way, we define the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions as a product of $(\overline{\Omega}^i, \overline{\mathcal{F}}^i, \overline{\mathbb{P}}^i; \overline{\mathbb{F}}^i)_{i=0}^N$.

Throughout the work, the symbol L and L_ϖ denote given positive constants, the symbol C a general positive constant which may change line by line. For a given constant $T > 0$, we use the following notation for frequently encountered spaces:

- $\mathbb{L}^2(\mathcal{G}; \mathbb{R}^d)$ denotes the set of $\mathbb{R}^d$-valued $\mathcal{G}$-measurable square integrable random variables.
- $\mathbb{S}^2(\mathbb{G}; \mathbb{R}^d)$ is the set of $\mathbb{R}^d$-valued $\mathbb{G}$-adapted continuous processes $\boldsymbol{X}$ satisfying

$$||X||_{\mathbb{S}^2} := \mathbb{E}\big[\sup_{t\in[0,T]} |X_t|^2\big]^{\frac{1}{2}} < \infty \ .$$

- $\mathbb{H}^2(\mathbb{G}; \mathbb{R}^d)$ is the set of $\mathbb{R}^d$-valued $\mathbb{G}$-progressively measurable processes $\boldsymbol{Z}$ satisfying

$$||Z||_{\mathbb{H}^2} := \mathbb{E}\Big[\Big(\int_0^T |Z_t|^2 dt\Big)\Big]^{\frac{1}{2}} < \infty \ .$$

- $\mathcal{L}(X)$ denotes the law of a random variable X.
- $\mathcal{P}(\mathbb{R}^d)$ is the set of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.
- $\mathcal{P}_p(\mathbb{R}^d)$ with $p \geq 1$ is the subset of $\mathcal{P}(\mathbb{R}^d)$ with finite p-th moment; i.e., the set of $\mu \in \mathcal{P}(\mathbb{R}^d)$ satisfying

$$M_p(\mu) := \Big(\int_{\mathbb{R}^d} |x|^p \mu(dx)\Big)^{\frac{1}{p}} < \infty \ .$$

We always assign $\mathcal{P}_p(\mathbb{R}^d)$ with $(p \geq 1)$ the p-Wasserstein distance W_p, which makes $\mathcal{P}_p(\mathbb{R}^d)$ a complete separable metric space. It is defined by, for any $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$,

$$W_p(\mu, \nu) := \inf_{\pi\in\Pi_p(\mu,\nu)} \Big[\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \pi(dx, dy)\Big)^{\frac{1}{p}}\Big] \tag{2.1}$$

where $\Pi_p(\mu, \nu)$ denotes the set of probability measures in $\mathcal{P}_p(\mathbb{R}^d \times \mathbb{R}^d)$ with marginals μ and ν. For more details, see Chapter 5 in [4].

- $m(\mu)$ denotes the expectation with respect to $\mu \in \mathcal{P}(\mathbb{R}^d)$, i.e.

$$m(\mu) := \int_{\mathbb{R}^d} x\mu(dx).$$

We frequently omit the arguments such as $(\mathbb{G}, \mathbb{R}^d)$ in the above definitions when there is no confusion from the context.

3 Market Clearing Equilibrium among Finite Agents

Our first goal is to prove the existence of the unique market clearing equilibrium for a stylized model of securities market and its characterization by the system of FBSDEs. The securities market we are interested in is basically the same as the one studied in the accompanying paper [19]. In the market, n types of securities are continuously traded via the securities exchange participated by the N security firms indexed by $i = 1, \cdots, N$. Every security firm, we will call it as an *agent*, is supposed to have many individual clients who cannot directly access to the exchange. In addition to the common systemic shocks, each agent faces the idiosyncratic shocks and the order-flow from its individual clients. In this environment, each agent $i \in \{1, \cdots, N\}$ tries to solve the optimization problem:

$$\inf_{\boldsymbol{\alpha}^i \in \mathbb{A}^i} J^i(\boldsymbol{\alpha}^i) \tag{3.1}$$

with the cost functional

$$J^i(\boldsymbol{\alpha}^i) := \mathbb{E}\Big[\int_0^T f_i(t, X_t^i, \alpha_t^i, \varpi_t, c_t^0, c_t^i)dt + g_i(X_T^i, \varpi_T, c_T^0, c_T^i)\Big],$$

subject to the dynamic constraint describing the time evolution of its securities' position:

$$dX_t^i = \Big(\alpha_t^i + l_i(t, \varpi_t, c_t^0, c_t^i)\Big)dt + \sigma_i^0(t, c_t^0, c_t^i)dW_t^0 + \sigma_i(t, c_t^0, c_t^i)dW_t^i, \quad t \in [0, T]$$

with $X_0^i = \xi^i$. Here, $\xi^i \in \mathbb{L}^2(\overline{\mathcal{F}}_0^i; \mathbb{R}^n)$ denotes the size of the initial position, which is assumed to have the common law for every $1 \leq i \leq N$. $(\varpi_t)_{t\in[0,T]} \in \mathbb{H}^2(\mathbb{F}; \mathbb{R}^n)$ denotes the market price process of n securities. One of our main goals is to derive the price process $(\varpi_t)_{t\in[0,T]}$ *endogenously* so that it achieves a market clearing equilibrium among those agents. $(c_t^0)_{t\geq 0} \in \mathbb{H}^2(\overline{\mathbb{F}}^0; \mathbb{R}^n)$ denotes the coupon payments from the securities or the market news affecting all the agents, while $(c_t^i)_{t\geq 0} \in \mathbb{H}^2(\overline{\mathbb{F}}^i; \mathbb{R}^n)$ denotes some idiosyncratic shocks affecting only the ith agent. Moreover, $(c_t^i)_{t\geq 0}$ are also assumed to have the common law for all $1 \leq i \leq N$. The terms involving $(l_i, \sigma_i^0, \sigma_i)$ denote the order-flow to the ith agent from its individual clients through the over-the-counter (OTC) market. Each agent controls $(\alpha_t^i)_{t\in[0,T]}$, which is an $\mathbb{R}^n$-valued process denoting the trading speed of the n securities via the exchange. Note that, in addition to the random initial states $(\xi^i)_{i=1}^N$, we have d_0-dimensional common noise W^0 and N d-dimensional idiosyncratic noises $(W^i)_{i=1}^N$. Since we impose no restriction on the size among (n, d_0, d, N), we have an incomplete securities market in general.

If the number of agents N is sufficiently large, it is natural to assume that each agent consider itself as a price taker. This means that each agent tries to solve the optimization problem by treating $(\varpi_t)_{t\geq 0}$ as an exogenous process. We firstly solve this individual optimization problem. We then use the result to search the market price process of the n securities which achieves the market clearing equilibrium among all the agents $1 \leq i \leq N$ who consider themselves as price takers. [1] We set the space of admissible strategies as $\mathbb{A}^i = \mathbb{H}^2(\mathbb{F}; \mathbb{R}^n)$. This means that each agent has the perfect information.

Remark 3.1. *Ideally, we would like to restrict the information set available to each agent i to the filtration $(\sigma\{\varpi_s : s \leq t\} \vee \mathcal{F}_t^i)_{t\geq 0}$. We postpone tackling this difficult problem for future research. Interestingly however, through the investigation of the mean-field limit of the market equilibrium in later sections, we shall observe that the information set of each agent i can be essentially restricted to $\mathbb{F}^i$ for large N.*

[1] In fact, most of the macroeconomic research is done under the price takers' equilibrium.

3.1 Individual optimization problem

For each agent i, let us introduce the cost functions: $f_i : [0,T] \times (\mathbb{R}^n)^5 \to \mathbb{R}$, $g_i : (\mathbb{R}^n)^4 \to \mathbb{R}$, $\overline{f}_i : [0,T] \times (\mathbb{R}^n)^4 \to \mathbb{R}$ and finally $\overline{g}_i : (\mathbb{R}^n)^3 \to \mathbb{R}$, which are measurable functions such that

$$
\begin{aligned}
f_i(t,x,\alpha,\varpi,c^0,c) &:= \langle \varpi, \alpha \rangle + \frac{1}{2}\langle \alpha, \Lambda\alpha \rangle + \overline{f}_i(t,x,\varpi,c^0,c), \\
g_i(x,\varpi,c^0,c) &:= -\delta\langle \varpi, x \rangle + \overline{g}_i(x,c^0,c).
\end{aligned}
$$

The following assumptions are fundamental to guarantee the unique solvability of the optimization problem for each agent.

Assumption 3.1. *Uniformly in $1 \le i \le N$, we assume the following conditions:*
(i) Λ is a positive definite $n \times n$ symmetric matrix with $\underline{\lambda}|\theta|^2 \le \langle \theta, \Lambda\theta \rangle \le \overline{\lambda}|\theta|^2$ for any $\theta \in \mathbb{R}^n$ where $0 < \underline{\lambda} \le \overline{\lambda}$ are some constants.
(ii) For any $(t,x,\varpi,c^0,c) \in [0,T] \times (\mathbb{R}^n)^4$,

$$
|\overline{f}_i(t,x,\varpi,c^0,c)| + |\overline{g}_i(x,c^0,c)| \le L(1 + |x|^2 + |\varpi|^2 + |c^0|^2 + |c|^2) \ .
$$

(iii) $\overline{f}_i$ and $\overline{g}_i$ are continuously differentiable in x and, for any $(t,x,x',\varpi,c^0,c) \in [0,T] \times (\mathbb{R}^n)^5$,

$$
|\partial_x \overline{f}_i(t,x',\varpi,c^0,c) - \partial_x \overline{f}_i(t,x,\varpi,c^0,c)| + |\partial_x \overline{g}_i(x',c^0,c) - \partial_x \overline{g}_i(x,c^0,c)| \le L|x'-x| \ ,
$$

and $|\partial_x \overline{f}_i(t,x,\varpi,c^0,c)| + |\partial_x \overline{g}_i(x,c^0,c)| \le L(1 + |x| + |\varpi| + |c^0| + |c|)$.
(iv)The functions $\overline{f}_i$ and $\overline{g}_i$ are convex in x in the sense that

$$
\begin{aligned}
&\overline{f}_i(t,x',\varpi,c^0,c) - \overline{f}_i(t,x,\varpi,c^0,c) - \langle x'-x, \partial_x \overline{f}_i(t,x,\varpi,c^0,c) \rangle \ge \frac{\gamma^f}{2}|x'-x|^2 \ , \\
&\overline{g}_i(x',c^0,c) - \overline{g}_i(x,c^0,c) - \langle x'-x, \partial_x \overline{g}_i(x,c^0,c) \rangle \ge \frac{\gamma^g}{2}|x'-x|^2 \ ,
\end{aligned}
$$

for any $(t,x,x',\varpi,c^0,c) \in [0,T] \times (\mathbb{R}^n)^5$ with some constants $\gamma^f, \gamma^g \ge 0$.
(v) $l_i : [0,T] \times (\mathbb{R}^n)^3 \to \mathbb{R}^n$, $\sigma_i^0 : [0,T] \times (\mathbb{R}^n)^2 \to \mathbb{R}^{n \times d_0}$, and $\sigma_i : [0,T] \times (\mathbb{R}^n)^2 \to \mathbb{R}^{n \times d}$ are the measurable functions satisfying, for any $(t,\varpi,c^0,c) \in [0,T] \times (\mathbb{R}^n)^3$,

$$
|l_i(t,\varpi,c^0,c)| + |\sigma_i^0(t,c^0,c)| + |\sigma_i(t,c^0,c)| \le L(1 + |\varpi| + |c^0| + |c|).
$$

(vi) $\delta \in [0,1)$ is a given constant.

Remark 3.2. *Note that the condition (iv) in the above assumptions implies*

$$
\begin{aligned}
&\big\langle x'-x, \partial_x \overline{f}_i(t,x',\varpi,c^0,c) - \partial_x \overline{f}_i(t,x,\varpi,c^0,c) \big\rangle \ge \gamma^f |x'-x|^2, \\
&\big\langle x'-x, \partial_x \overline{g}_i(x',c^0,c) - \partial_x \overline{g}_i(x,c^0,c) \big\rangle \ge \gamma^g |x'-x|^2,
\end{aligned}
$$

which is frequently used in the following analyses.

Economic interpretation of each term is as follows; the first term $\langle \varpi, \alpha \rangle$ of f_i denotes the direct cost incurred by the sales and purchase of the securities via the exchange and the second term $\frac{1}{2}\langle \alpha, \Lambda\alpha \rangle$ is some fee to be paid to the exchange based on the agent's trading speed. $-\delta\langle \varpi, x \rangle$ denotes the mark-to-market value at the terminal time T with some discount factor $\delta < 1$. The other terms $\overline{f}_i, \overline{g}_i$ denote the running as well as the terminal costs depending on the storage level of the securities and their price. (Λ, δ) is assumed to be common for all the agents to make the problem simpler.

The associated (reduced) Hamiltonian [2] $H_i : [0,T] \times (\mathbb{R}^n)^6 \to \mathbb{R}$

$$H_i(t,x,y,\alpha,\varpi,c^0,c) := \langle y, \alpha + l_i(t,\varpi,c^0,c)\rangle + f_i(t,x,\alpha,\varpi,c^0,c)$$

has a unique minimizer

$$\widehat{\alpha}(y,\varpi) := -\overline{\Lambda}(y+\varpi) \tag{3.2}$$

where $\overline{\Lambda} := \Lambda^{-1}$. The adjoint equation for the ith agent arising from the stochastic maximum principle is then given by

$$\begin{aligned} dX_t^i &= \Big(\widehat{\alpha}(Y_t^i,\varpi_t) + l_i(t,\varpi_t,c_t^0,c_t^i)\Big)dt + \sigma_i^0(t,c_t^0,c_t^i)dW_t^0 + \sigma_i(t,c_t^0,c_t^i)dW_t^i, \\ dY_t^i &= -\partial_x \overline{f}_i(t,X_t^i,\varpi_t,c_t^0,c_t^i)dt + Z_t^{i,0}dW_t^0 + \sum_{j=1}^N Z_t^{i,j}dW_t^j, \end{aligned} \tag{3.3}$$

with $X_0^i = \xi^i$ and $Y_T^i := \partial_x g_i(X_T^i,\varpi_T,c_T^0,c_T^i)$.

Theorem 3.1. *Let Assumption 3.1 be in force. Then, for a given $(\varpi_t)_{t\in[0,T]} \in \mathbb{H}^2(\mathbb{F};\mathbb{R}^n)$, the problem* (3.1) *for each agent $1 \le i \le N$ is uniquely characterized by the FBSDE* (3.3) *which is strongly solvable with a unique solution $(X^i, Y^i, Z^{i,0}, (Z^{i,j})_{j=1}^N) \in \mathbb{S}^2(\mathbb{F};\mathbb{R}^n) \times \mathbb{S}^2(\mathbb{F};\mathbb{R}^n) \times \mathbb{H}^2(\mathbb{F};\mathbb{R}^{n\times d_0}) \times (\mathbb{H}^2(\mathbb{F};\mathbb{R}^{n\times d}))^N$.*

Proof. This is essentially the same as Theorem 3.1 in [19]. Since the cost functions are jointly convex with (x,α) and strictly convex in α, the problem is the special situation investigated in Section 1.4.4 in [5]. It can be proved in a similar way as Theorem 1.60 in the same reference. One can also prove via Peng-Wu's method [41]. In fact, the method will be applied to a much more complex situation below. □

3.2 Market clearing equilibrium among N agents

From Theorem 3.1, the optimal trading strategy of the agent i for a given $(\varpi_t)_{t\in[0,T]}$ is

$$\widehat{\alpha}_t^i := -\overline{\Lambda}(Y_t^i + \varpi_t), \quad t \in [0,T].$$

Since the market clearing requires $\sum_{i=1}^N \widehat{\alpha}_t^i = 0$, the market price needs to satisfy

$$\varpi_t = -\frac{1}{N}\sum_{i=1}^N Y_t^i, \quad t \in [0,T]. \tag{3.4}$$

[2]Since σ_i^0, σ_i are independent of the control α^i and also the state x^i, it suffices to use the reduced Hamiltonian for the adjoint equation.

This means that the market clearing equilibrium is realized if the following $1 \le i \le N$ coupled system of FBSDEs have a solution;

$$
\begin{aligned}
dX_t^i &:= \Big\{ \widehat{\alpha}\Big(Y_t^i, -\frac{1}{N}\sum_{j=1}^N Y_t^j\Big) + l_i\Big(t, -\frac{1}{N}\sum_{j=1}^N Y_t^j, c_t^0, c_t^i\Big)\Big\} dt \\
&\qquad + \sigma_i^0(t, c_t^0, c_t^i) dW_t^0 + \sigma_i(t, c_t^0, c_t^i) dW_t^i, \\
dY_t^i &= -\partial_x \overline{f}_i\Big(t, X_t^i, -\frac{1}{N}\sum_{j=1}^N Y_t^j, c_t^0, c_t^i\Big) dt + Z_t^{i,0} dW_t^0 + \sum_{j=1}^N Z_t^{i,j} dW_t^j,
\end{aligned} \tag{3.5}
$$

for $t \in [0, T]$ with

$$
\begin{aligned}
X_0^i &= \xi^i, \\
Y_T^i &= \frac{\delta}{1-\delta}\frac{1}{N}\sum_{j=1}^N \partial_x \overline{g}_j(X_T^j, c_T^0, c_T^j) + \partial_x \overline{g}_i(X_T^i, c_T^0, c_T^i).
\end{aligned} \tag{3.6}
$$

Let us mention about the terminal condition. Since we have (3.4), Y_T^i must satisfy

$$
Y_T^i = \delta \frac{1}{N}\sum_{j=1}^N Y_T^j + \partial_x \overline{g}_i(X_T^i, c_T^0, c_T^i).
$$

Summing over $1 \le i \le N$, we obtain the expression $\frac{1}{N}\sum_{j=1}^N Y_T^j$. Substituting the result into the above terminal condition, we get the desired result.

Theorem 3.2. *If there exists a solution to the N-coupled system of FBSDEs* (3.5) *with* (3.6)*, then the price process defined by* (3.4) *achieves market clearing equilibrium among the N agents who consider themselves as price takers.*

Proof. Suppose that the N-coupled system of FBSDEs (3.5) with (3.6) has in fact a solution. Let us set a market price process as (3.4) by using the solution of the system of FBSDEs, $(Y^i)_{i=1}^N$. Due to the uniqueness result in Theorem 3.1, the solution to the associated adjoint equation (3.3) for the individual optimization problem is in fact given by the same Y^i, $1 \le i \le N$. This means that (3.4) actually provides a market clearing price of the n securities among the N agents. $\square$

We now introduce a new set of assumptions to prove the existence of the solution to (3.5).

Assumption 3.2. *(i) For any $(t, x, \varpi, \varpi', c^0, c) \in [0, T] \times (\mathbb{R}^n)^5$,*

$$
|\partial_x \overline{f}_i(t, x, \varpi, c^0, c) - \partial_x \overline{f}_i(t, x, \varpi', c^0, c)| + |l_i(t, \varpi, c^0, c) - l_i(t, \varpi', c^0, c)| \le L_\varpi |\varpi - \varpi'|,
$$

for every $1 \le i \le N$.
(ii) For any $(t, c^0) \in [0, T] \times \mathbb{R}^n$ and $(x^i, x^{i\prime}, c^i) \in (\mathbb{R}^n)^3, 1 \le i \le N$, the functions $(l_i)_{i=1}^N$ satisfy with some $\gamma^l > 0$

$$
\sum_{i=1}^N \Big\langle l_i\Big(t, \frac{1}{N}\sum_{j=1}^N x^j, c^0, c^i\Big) - l_i\Big(t, \frac{1}{N}\sum_{j=1}^N x^{j\prime}, c^0, c^i\Big), x^i - x^{i\prime} \Big\rangle \ge N\gamma^l \Big| \frac{1}{N}\sum_{i=1}^N (x^i - x^{i\prime}) \Big|^2.
$$

(iii) There exists a strictly positive constant γ satisfying

$$0 < \gamma \le \Big(\gamma^f - \frac{L_\varpi^2}{4\gamma^l}\Big) \wedge \gamma^g \ .$$

Moreover, the functions $(\overline{g}_i)_{i=1}^N$ satisfy for any $c^0 \in \mathbb{R}^n$ and $(x^i, x^{i\prime}, c^i) \in (\mathbb{R}^n)^3, 1 \le i \le N$,

$$\frac{\delta}{1-\delta}\sum_{i=1}^N \Big\langle \frac{1}{N}\sum_{j=1}^N \partial_x \overline{g}_j(x^j, c^0, c^j) - \frac{1}{N}\sum_{j=1}^N \partial_x \overline{g}_j(x^{j\prime}, c^0, c^j), x^i - x^{i\prime}\Big\rangle \ge (\gamma - \gamma^g)\sum_{i=1}^N |x^i - x^{i\prime}|^2.$$

Remark 3.3. *In economic terms, the monotone condition (ii) can be interpreted in a very natural way. It basically tells that the demand for the securities from the OTC clients of the agents decreases when the market price rises. Let us provide the simplest example of the functions $(l_i)_i$ that satisfy (ii); assume that l_i has a separable form $l_i(t, x, c^0, c^i) = h(t, x) + h_i(t, c^0, c^i)$ and also that the common function h is strictly monotone in x. Then, one can easily check that (ii) is satisfied. The third condition can be clearly satisfied when $(\partial_x \overline{g}_i)_i$ have a similar structure.*

For notational convenience for later analysis, let us introduce the following functions: $B_i : [0,T] \times \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \times (\mathbb{R}^n)^2 \to \mathbb{R}^n$, $F_i : [0,T] \times \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \times (\mathbb{R}^n)^2 \to \mathbb{R}^n$ and $G_i : \mathcal{P}(\mathbb{R}^n) \times (\mathbb{R}^n)^3 \to \mathbb{R}^n$, $1 \le i \le N$ by

$$\begin{aligned}
&B_i(t, x, \mu, c^0, c) := \widehat{\alpha}(x, -m(\mu)) + l_i(t, -m(\mu), c^0, c),\\
&F_i(t, x, \mu, c^0, c) := -\partial_x \overline{f}_i(t, x, -m(\mu), c^0, c),\\
&G_i(\mu, x, c^0, c) := \frac{\delta}{1-\delta} m(\mu) + \partial_x \overline{g}_i(x, c^0, c),
\end{aligned} \tag{3.7}$$

for any $(t, x, \mu, c^0, c) \in [0,T] \times \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \times (\mathbb{R}^n)^2$.

Theorem 3.3. *Let Assumptions 3.1 and 3.2 be in force. Then the N-coupled system of FBSDEs (3.5) with (3.6) has a unique strong solution $(X^i, Y^i, Z^{i,0}, (Z^{i,j})_{j=1}^N) \in \mathbb{S}^2(\mathbb{F}; \mathbb{R}^n) \times \mathbb{S}^2(\mathbb{F}; \mathbb{R}^n) \times \mathbb{H}^2(\mathbb{F}; \mathbb{R}^{n\times d_0}) \times (\mathbb{H}^2(\mathbb{F}; \mathbb{R}^{n\times d}))^N$, $1 \le i \le N$.*

Proof. We can prove the claim by a simple modification of Theorem 6.2 in [19]. First, we make the following hypothesis: *there exists some constant $\varrho \in [0,1)$ such that for any $I^{b,i}, I^{f,i} \in \mathbb{H}^2(\mathbb{F}; \mathbb{R}^n)$ and for any $\eta^i \in \mathbb{L}^2(\mathcal{F}_T; \mathbb{R}^n)$, there exists a unique strong solution $(x^{\varrho,i}, y^{\varrho,i}, z^{\varrho,i,0}, (z^{\varrho,i,j})_{j=1}^N) \in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{H}^2 \times (\mathbb{H}^2)^N$, $1 \le i \le N$ to the N-coupled system of FBSDEs:*

$$\begin{aligned}
dx_t^{\varrho,i} &= \big(\varrho B_i(t, y_t^{\varrho,i}, \mu_t^{\varrho,N}, c_t^0, c_t^i) + I_t^{b,i}\big)dt + \sigma_i^0(t, c_t^0, c_t^i)dW_t^0 + \sigma_i(t, c_t^0, c_t^i)dW_t^i,\\
dy_t^{\varrho,i} &= -\big((1-\varrho)\gamma x_t^{\varrho,i} - \varrho F_i(t, x_t^{\varrho,i}, \mu_t^{\varrho,N}, c_t^0, c_t^i) + I_t^{f,i}\big)dt + z_t^{\varrho,i,0}dW_t^0 + \sum_{j=1}^N z_t^{\varrho,i,j}dW_t^j,
\end{aligned}$$

for $t \in [0,T]$ with $x_0^{\varrho,i} = \xi^i$ and $y_T^{\varrho,i} = \varrho G_i(\mu_g^{\varrho,N}, x_T^{\varrho,i}, c_T^0, c_T^i) + (1-\varrho)x_T^{\varrho,i} + \eta^i$. Here,

$$\mu_t^{\varrho,N} := \frac{1}{N}\sum_{i=1}^N \delta_{y_t^{\varrho,i}}, \quad \mu_g^{\varrho,N} := \frac{1}{N}\sum_{i=1}^N \delta_{\partial_x \overline{g}_i(x_T^{\varrho,i}, c_T^i, c_T^i)}$$

denote the empirical measures. Notice that the system reduces to the N decoupled FBSDEs when $\varrho = 0$. Hence, the hypothesis trivially holds for $\varrho = 0$.

Now, for some constant $\zeta \in (0,1)$, we define a map

$$\begin{aligned} &\big(\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{H}^2 \times (\mathbb{H}^2)^N\big)^N \ni \big(x^i, y^i, z^{i,0}, (z^{i,j})_{j=1}^N\big)_{i=1}^N \\ &\qquad \mapsto \big(X^i, Y^i, Z^{i,0}, (Z^{i,j})_{j=1}^N\big)_{i=1}^N \in \big(\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{H}^2 \times (\mathbb{H}^2)^N\big)^N \end{aligned} \tag{3.8}$$

by

$$\begin{aligned} dX_t^i &= \big[\varrho B_i(t, Y_t^i, \mu_t^N, c_t^0, c_t^i) + \zeta B_i(t, y_t^i, \nu_t^N, c_t^0, c_t^i) + I_t^{b,i}\big]dt \\ &\qquad + \sigma_i^0(t, c_t^0, c_t^i) dW_t^0 + \sigma_i(t, c_t^0, c_t^i) dW_t^i, \\ dY_t^i &= -\Big[(1-\varrho)\gamma X_t^i - \varrho F_i(t, X_t^i, \mu_t^N, c_t^0, c_t^i) + \zeta\big(-\gamma x_t^i - F_i(t, x_t^i, \nu_t^N, c_t^0, c_t^i)\big) + I_t^{f,i}\Big]dt \\ &\qquad + Z_t^{i,0} dW_t^0 + \sum_{j=1}^N Z_t^{i,j} dW_t^j, \end{aligned}$$

with $X_0^i = \xi$ and $Y_T^i = \varrho G_i(\mu_g^N, X_T^i, c_T^0, c_T^i) + (1-\varrho)X_T^i + \zeta\big(G_i(\nu_g^N, x_T^i, c_T^0, c_T^i) - x_T^i\big) + \eta^i$. Here, the measure arguments are defined by

$$\begin{aligned} \mu_t^N &:= \frac{1}{N}\sum_{i=1}^N \delta_{Y_t^i}, \qquad \nu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{y_t^i}, \\ \mu_g^N &:= \frac{1}{N}\sum_{i=1}^N \delta_{\partial_x \overline{g}_i(X_T^i, c_T^0, c_T^i)}, \quad \nu_g^N := \frac{1}{N}\sum_{i=1}^N \delta_{\partial_x \overline{g}_i(x_T^i, c_T^0, c_T^i)}. \end{aligned}$$

Thanks for the hypothesis, there exists a unique solution $\big(X^i, Y^i, Z^{i,0}, (Z^{i,j})_{j=1}^N\big)_{i=1}^N$ and hence the map (3.8) is well-defined.

Consider the two set of inputs $\big(x^i, y^i, z^{i,0}, (z^{i,j})_{j=1}^N\big)_{i=1}^N$ and $\big(x^{i\prime}, y^{i\prime}, z^{i,0\prime}, (z^{i,j\prime})_{j=1}^N\big)_{i=1}^N$, and then denote the corresponding solution to the previous FBSDEs by $\big(X^i, Y^i, Z^{i,0}, (Z^{i,j})_{j=1}^N\big)_{i=1}^N$ and $\big(X^{i\prime}, Y^{i\prime}, Z^{i,0\prime}, (Z^{i,j\prime})_{j=1}^N\big)_{i=1}^N$, respectively. Put $\Delta X^i := X^i - X^{i\prime}, \Delta Y^i := Y^i - Y^{i\prime}$, etc. Since $\Delta X_0^i = 0$, a simple application of Itô-formula yields

$$\begin{aligned} \sum_{i=1}^N \mathbb{E}\big[\langle \Delta X_T^i, \Delta Y_T^i \rangle\big] &= -(1-\varrho)\gamma \mathbb{E}\int_0^T \sum_{i=1}^N |\Delta X_t^i|^2 dt \\ &\quad + \varrho \mathbb{E}\int_0^T \sum_{i=1}^N \big\langle B_i(t, Y_t^i, \mu_t^N, c_t^0, c_t^i) - B_i(t, Y_t^{i\prime}, \mu_t^{\prime N}, c_t^0, c_t^i), \Delta Y_t^i \big\rangle dt \\ &\quad + \varrho \mathbb{E}\int_0^T \sum_{i=1}^N \big\langle F_i(t, X_t^i, \mu_t^N, c_t^0, c_t^i) - F_i(t, X_t^{i\prime}, \mu_t^{\prime N}, c_t^0, c_t^i), \Delta X_t^i \big\rangle dt \\ &\quad + \zeta \mathbb{E}\int_0^T \sum_{i=1}^N \big\langle B_i(t, y_t^i, \nu_t^N c_t^0, c_t^i) - B_i(t, y_t^{i\prime}, \nu_t^{\prime N}, c_t^0, c_t^i), \Delta Y_t^i \big\rangle dt \\ &\quad + \zeta \mathbb{E}\int_0^T \sum_{i=1}^N \big\langle \gamma \Delta x_t^i + F_i(t, x_t^i, \nu_t, c_t^0, c_t^i) - F_i(t, x_t^{i\prime}, \nu_t^{\prime N}, c_t^0, c_t^i), \Delta X_t^i \big\rangle dt\ . \end{aligned}$$

Using the convexity as well as the monotone conditions in Assumptions 3.1 and 3.2, and also the inequality

$$-\sum_{i=1}^{N}\Big\langle \overline{\Lambda}\Big(\Delta Y_t^i-\frac{1}{N}\sum_{j=1}^{N}\Delta Y_t^j\Big),\Delta Y_t^i\Big\rangle\le 0,$$

we obtain, with some constant C independent of (ϱ,N), that

$$\begin{aligned}\sum_{i=1}^{N}\mathbb{E}\big[\langle\Delta X_T^i,\Delta Y_T^i\rangle\big]&\le-\gamma\mathbb{E}\int_0^T\sum_{i=1}^{N}|\Delta X_t^i|^2dt\\&+\zeta C\mathbb{E}\int_0^T\sum_{i=1}^{N}\Big[\big(|\Delta y_t^i|+|\Delta\overline{y}_t|\big)|\Delta Y_t^i|+\big(|\Delta x_t^i|+|\Delta\overline{y}_t|\big)|\Delta X_t^i|\Big]dt,\end{aligned}$$

where $\Delta\overline{y}_t:=\frac{1}{N}\sum_{i=1}^{N}\Delta y_t^i$. On the other hand, we get from the terminal condition,

$$\begin{aligned}\sum_{i=1}^{N}\mathbb{E}\big[\langle\Delta X_T^i,\Delta Y_T^i\rangle\big]&=\varrho\mathbb{E}\sum_{i=1}^{N}\big\langle G_i(\mu_g^N,X_T^i,c_T^0,c_T^i)-G_i(\mu_g'^N,X_T^{i\prime},c_T^0,c_T^i),\Delta X_T^i\big\rangle\\&+(1-\varrho)\mathbb{E}\sum_{i=1}^{N}\big\langle\Delta X_T^i,\Delta X_T^i\big\rangle+\zeta\mathbb{E}\sum_{i=1}^{N}\big\langle G_i(\nu_g^N,x_T^i,c_T^0,c_T^i)-G_i(\nu_g'^N,x_T^{i\prime},c_T^0,c_T^i),\Delta X_T^i\big\rangle\\&\ge(\varrho\gamma+(1-\varrho))\mathbb{E}\Big[\sum_{i=1}^{N}|\Delta X_T^i|^2\Big]-\zeta C\mathbb{E}\Big[\sum_{i=1}^{N}\big(|\Delta x_T^i|+\overline{|\Delta x_T|}\big)|\Delta X_T^i|\Big],\end{aligned}$$

where $\overline{|\Delta x_T|}:=\frac{1}{N}\sum_{i=1}^{N}|\Delta x_T^i|$. With $\gamma_c:=\min(1,\gamma)$, we have $0<\gamma_c\le\varrho\gamma+(1-\varrho)$, and hence the above two estimates give

$$\begin{aligned}\gamma_c\sum_{i=1}^{N}\mathbb{E}\Big[|\Delta X_T^i|^2+\int_0^T|\Delta X_t^i|^2dt\Big]&\le\zeta C\sum_{i=1}^{N}\mathbb{E}\Big[(\overline{|\Delta x_T|}+|\Delta x_T^i|)|\Delta X_T^i|\Big]\\&+\zeta C\mathbb{E}\int_0^T\sum_{i=1}^{N}\Big[(|\Delta y_t^i|+|\Delta\overline{y}_t|)|\Delta Y_t^i|+\big(|\Delta x_t^i|+|\Delta\overline{y}_t|\big)|\Delta X_t^i|\Big]dt.\end{aligned}$$

Using Young's inequality, we obtain

$$\sum_{i=1}^{N}\mathbb{E}\Big[|\Delta X_T^i|^2+\int_0^T|\Delta X_t^i|^2dt\Big]\le\zeta C\sum_{i=1}^{N}\mathbb{E}\Big[|\Delta x_T^i|^2+\int_0^T\Big(\big[|\Delta x_t^i|^2+|\Delta y_t^i|^2\big]+|\Delta Y_t^i|^2\Big)dt\Big]\tag{3.9}$$

Let us now treat $(X^i,X^{i\prime})_{i=1}^N$ as the exogenous inputs. Then the standard stability result for the Lipschitz BSDEs (see, for example, Theorem 4.2.3 in [44]) implies

$$
\begin{aligned}
&\sum_{i=1}^N \mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta Y_t^i|^2+\int_0^T|\Delta Z_t^{i,0}|^2dt+\sum_{j=1}^N\int_0^T|\Delta Z_t^{i,j}|^2dt\Big]\\
&\le C\sum_{i=1}^N\mathbb{E}\Big[|\Delta X_T^i|^2+\int_0^T|\Delta X_t^i|^2dt\Big]+\zeta C\sum_{i=1}^N\mathbb{E}\Big[|\Delta x_T^i|^2+\int_0^T\big(|\Delta x_t^i|^2+|\Delta y_t^i|^2\big)dt\Big].
\end{aligned}
$$

Using (3.9) and small ζ, we obtain

$$
\begin{aligned}
&\sum_{i=1}^N \mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta Y_t^i|^2+\int_0^T|\Delta Z_t^{i,0}|^2dt+\sum_{j=1}^N\int_0^T|\Delta Z_t^{i,j}|^2dt\Big]\\
&\qquad\le \zeta C\sum_{i=1}^N\mathbb{E}\Big[|\Delta x_T^i|^2+\int_0^t\big(|\Delta x_t^i|^2+|\Delta y_t^i|^2\big)dt\Big]. \qquad (3.10)
\end{aligned}
$$

Similarly, by treating $(Y^i,Y^{i\prime})_{i=1}^N$ as the exogenous inputs, the standard stability result for the Lipschitz SDEs gives

$$
\sum_{i=1}^N\mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta X_T^i|\Big]\le\zeta C\sum_{i=1}^N\mathbb{E}\int_0^T|\Delta y_t^i|^2dt+C\sum_{i=1}^N\mathbb{E}\int_0^T|\Delta Y_t^i|^2dt. \qquad (3.11)
$$

Therefore, from (3.10) and (3.11), we obtain

$$
\begin{aligned}
&\sum_{i=1}^N \mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta X_t^i|^2+\sup_{t\in[0,T]}|\Delta Y_t^i|^2+\int_0^T|\Delta Z_t^{i,0}|^2dt+\sum_{j=1}^N\int_0^T|\Delta Z_t^{i,j}|^2dt\Big]\\
&\le\zeta C\sum_{i=1}^N\mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta x_t^i|^2+\sup_{t\in[0,T]}|\Delta y_t^i|^2\Big].
\end{aligned}
$$

Thus for small $\zeta>0$, which can be taken independently from ϱ, the map (3.8) becomes a strict contraction. Hence the Banach fixed point theorem implies that the initial hypothesis holds for $(\varrho+\zeta)$. Repeating the procedures, we see the hypothesis holds with $\varrho=1$. This establishes the existence of a solution. The uniqueness is a direct result of the next stability estimate. □

Proposition 3.1. *Given two set of inputs $(\xi^i,c^0,c^i)_{i=1}^N$, $(\xi^{i\prime},c^{0\prime},c^{i\prime})_{i=1}^N$, and the coefficients functions $(l_i,\sigma_i^0,\sigma_i,f_i,g_i)_{i=1}^N$, $(l_i',\sigma_i^{0\prime},\sigma_i',f_i',g_i')$ satisfying Assumptions 3.1 and 3.2, let us denote the corresponding solutions to* (3.5) *with* (3.6) *by $(X^i,Y^i,Z^{i,0},(Z^{i,j})_{j=1}^N)_{i=1}^N$ and $(X^{i\prime},Y^{i\prime},Z^{i,0\prime},(Z^{i,j\prime})_{j=1}^N)_{i=1}^N$, respectively. Then, for $\Delta X^i:=X^i-X^{i\prime},\Delta Y^i:=Y^i-Y^{i\prime},\Delta Z^{i,j}:=Z^{i,j}-Z^{i,j\prime}$, $1\le i,j\le N$, we have*

$$
\begin{aligned}
&\sum_{i=1}^N \mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta X_t^i|^2+\sup_{t\in[0,T]}|\Delta Y_t^i|^2+\int_0^T\big(|\Delta Z_t^{i,0}|^2+\sum_{j=1}^N|\Delta Z_t^{i,j}|^2\big)dt\Big]\\
&\le C\sum_{i=1}^N\mathbb{E}\Big[|\Delta\xi^i|^2+|\overline{G}_i|^2+\int_0^T\big(|\overline{F}_i(t)|^2+|\overline{B}_i(t)|^2+|\overline{\sigma}_i^0(t)|^2+|\overline{\sigma}_i(t)|^2\big)dt\Big],
\end{aligned}
$$

with some constant C depending only on the Lipschitz constants, $\delta, \underline{\lambda}$ and γ. Here,

$$
\begin{aligned}
&\overline{B}_i(t) := B_i(t, Y_t^{i\prime}, \mu_t^{\prime N}, c_t^0, c_t^i) - B_i'(t, Y_t^{i\prime}, \mu_t^{\prime N}, c_t^{0\prime}, c_t^{i\prime}),\\
&\overline{F}_i(t) := F_i(t, X_t^{i\prime}, \mu_t^{\prime N}, c_t^0, c_t^i) - F_i'(t, X_t^{i\prime}, \mu_t^{\prime N}, c_t^{0\prime}, c_t^{i\prime}),\\
&\overline{G}_i := G_i(\mu_g^{\prime N}, X_T^{i\prime}, c_T^0, c_T^i) - G_i'(\mu_{g'}^{\prime N}, X_T^{i\prime}, c_T^{0\prime}, c_T^{i\prime}),\\
&(\overline{\sigma}_i^0, \overline{\sigma}_i)(t) := \big((\sigma_i^0, \sigma_i)(t, c_t^0, c_t^i) - (\sigma_i^{0\prime}, \sigma_i')(t, c_t^{0\prime}, c_t^{i\prime})\big),
\end{aligned}
$$

for $t \in [0,T]$ and $1 \le i \le N$. Here B_i', F_i' and G_i' are defined as (3.7) *with primed variables. The measure arguments are defined by $\mu_t^{\prime N} := \frac{1}{N}\sum_{i=1}^N \delta_{Y_t^{i\prime}}$, $\mu_g^{\prime N} := \frac{1}{N}\sum_{i=1}^N \delta_{\partial_x \overline{g}_i}(X_T^{i\prime}, c_T^0, c_T^i)$ and $\mu_{g'}^{\prime N} := \frac{1}{N}\sum_{i=1}^N \delta_{\partial_x \overline{g}_i'}(X_T^{i\prime}, c_T^{0\prime}, c_T^{i\prime})$.*

Proof. One can prove the claim exactly the same way as in Proposition 4.1 in [19]. □

Corollary 3.1. *Let Assumptions 3.1 and 3.2 be in force. Then the solution $(X^i, Y^i, Z^{i,0}, (Z^{i,j})_{j=1}^N)_{i=1}^N$ to the system of FBSDEs* (3.5) *with* (3.6) *satisfies*

$$
\begin{aligned}
&\sum_{i=1}^N \mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^i|^2 + \sup_{t\in[0,T]} |Y_t^i|^2 + \int_0^T \big(|Z_t^{i,0}|^2 + \sum_{j=1}^N |Z_t^{i,j}|^2\big) dt\Big]\\
&\le C \sum_{i=1}^N \mathbb{E}\Big[|\xi^i|^2 + |\partial_x \overline{g}_i(0, c_T^0, c_T^i)|^2\\
&\qquad\qquad + \int_0^T \big(|\partial_x \overline{f}_i(t, 0, 0, c_t^i, c_t)|^2 + |l_i(t, 0, c_t^0, c_t^i)|^2 + |(\sigma_i^0, \sigma_i)(t, c_t^0, c_t^i)|^2\big) dt\Big] \ ,
\end{aligned}
$$

where C is some constant depending only on the Lipschitz constants, $\delta, \underline{\lambda}$ and γ.

Proof. This is the immediate consequence of Proposition 3.1. See Corollary 4.1 in [19]. □

4 Strong Convergence to the Mean-Field Limit

Convergence among the homogeneous agents

Let us introduce a new set of coefficients $(\delta, \Lambda, l, \sigma^0, \sigma, \overline{f}, \overline{g})$ satisfying the following conditions.

Assumption 4.1. *(i) $(\delta, \Lambda, l, \sigma^0, \sigma, \overline{f}, \overline{g})$ satisfies Assumptions 3.1 and 3.2. Here, the conditions (ii) and (iii) in Assumption 3.2 are replaced by the N copies of l and $\partial_x \overline{g}$.*
(ii) For any $t \in [0,T]$, any random variables $x, x', c^0, c \in \mathbb{L}^2(\mathcal{F}; \mathbb{R}^n)$ and any sub-σ field $\mathcal{G} \subset \mathcal{F}$, the function l satisfies

$$
\mathbb{E}\Big[\big\langle l(t, \mathbb{E}[x|\mathcal{G}], c^0, c) - l(t, \mathbb{E}[x'|\mathcal{G}], c^0, c), x - x'\big\rangle\Big] \ge \gamma \mathbb{E}\Big[\mathbb{E}\big[x - x'|\mathcal{G}\big]^2\Big].
$$

(iii) For any random variables $x, x', c^0, c \in \mathbb{L}^2(\mathcal{F}; \mathbb{R}^n)$ and any sub-σ field $\mathcal{G} \subset \mathcal{F}$, the function $\overline{g}$ satisfies

$$
\frac{\delta}{1-\delta}\mathbb{E}\Big[\big\langle \mathbb{E}\big[\partial_x \overline{g}(x, c^0, c) - \partial_x \overline{g}(x', c^0, c)|\mathcal{G}\big], x - x'\big\rangle\Big] \ge \gamma \mathbb{E}\big[|x - x'|^2\big].
$$

Remark 4.1. *Note that the conditions (ii) and (iii) are natural generalization of those of Assumption 3.2 where they are given in terms of the empirical mean.*

Using these coefficient functions, consider the following conditional McKean-Vlasov FBSDEs:

$$
\begin{aligned}
d\overline{X}_t^i &= \Big(\widehat{\alpha}\big(\overline{Y}_t^i, -\mathbb{E}[\overline{Y}_t^i|\overline{\mathcal{F}}_t^0]\big) + l\big(t, -\mathbb{E}[\overline{Y}_t^i|\overline{\mathcal{F}}_t^0], c_t^0, c_t^i\big)\Big)dt + \sigma^0(t, c_t^0, c_t^i)dW_t^0 + \sigma(t, c_t^0, c_t^i)dW_t^i, \\
d\overline{Y}_t^i &= -\partial_x \overline{f}\big(t, \overline{X}_t^i, -\mathbb{E}[\overline{Y}_t^i|\overline{\mathcal{F}}_t^0], c_t^0, c_t^i\big)dt + \overline{Z}_t^{i,0}dW_t^0 + \overline{Z}_t^{i,i}dW_t^i, \qquad (4.1)
\end{aligned}
$$

for $t \in [0,T]$ with

$$
\begin{aligned}
\overline{X}_0^i &= \xi^i, \\
\overline{Y}_T^i &= \frac{\delta}{1-\delta}\mathbb{E}\big[\partial_x \overline{g}(\overline{X}_T^i, c_T^0, c_T^i)|\overline{\mathcal{F}}_T^0\big] + \partial_x \overline{g}(\overline{X}_T^i, c_T^0, c_T^i)\ . \qquad (4.2)
\end{aligned}
$$

We know the following result.

Theorem 4.1. *(Theorem 4.2 [19]) Let Assumption 4.1 be in force. Then, for each $1 \le i \le N$, there exists a unique strong solution $(\overline{X}^i, \overline{Y}^i, \overline{Z}^{i,0}, \overline{Z}^{i,i}) \in \mathbb{S}^2(\mathbb{F}^i;\mathbb{R}^n) \times \mathbb{S}^2(\mathbb{F}^i;\mathbb{R}^n) \times \mathbb{H}^2(\mathbb{F}^i;\mathbb{R}^{n\times d_0}) \times \mathbb{H}^2(\mathbb{F}^i;\mathbb{R}^{n\times d})$ to the FBSDE of conditional McKean-Vlasov type* (4.1) *with* (4.2).

Note that FBSDE (4.1) is decoupled for each $1 \le i \le N$ and that its solution has the same distribution. In particular, for given $\overline{\mathcal{F}}^0$, the solutions $(\overline{X}^i, \overline{Y}^i, \overline{Z}^{i,0}, \overline{Z}^{i,i}), 1 \le i \le N$ are independently and identically distributed. Because of this property, the quantities such as $\mathbb{E}[\overline{Y}_t^i|\overline{\mathcal{F}}_t^0]$ and $\mathbb{E}[\partial_x \overline{g}(\overline{X}_T^i, c_T^0, c_T^i)|\overline{\mathcal{F}}_T^0]$ are independent of the index i.

The FBSDE (4.1) has been the major object of the analysis in the accompanying work [19], in which we have found that the $\overline{\mathbb{F}}^0$-progressively measurable process

$$
\varpi_t^{\mathrm{MFG}} := -\mathbb{E}\big[\overline{Y}_t^i|\overline{\mathcal{F}}_t^0\big] = -\mathbb{E}\big[\overline{Y}_t^i|\overline{\mathcal{F}}^0\big], \qquad t \in [0,T]
$$

provides a good approximate of the equilibrium market price if the agents have the common coefficients as in Assumption 4.1. In particular, we have proved in Theorem 5.1 [19] that the process ϖ^{MFG} achieves the market clearing in the large N limit. The goal of this section is to prove the strong convergence of the N-agent equilibrium given by Theorem 3.2 and 3.3 to the above mean-field limit when the agents are homogeneous. Once this is done, we can study the stability relation of the market price for the heterogeneous agents relative to the mean-field limit ϖ^{MFG} with the help of Proposition 3.1.

The market equilibrium among homogeneous agents is specified by N-coupled FBSDEs (3.5) with common coefficient functions:

$$
\begin{aligned}
dX_t^i &:= \Big\{\widehat{\alpha}\Big(Y_t^i, -\frac{1}{N}\sum_{j=1}^N Y_t^j\Big) + l\Big(t, -\frac{1}{N}\sum_{j=1}^N Y_t^j, c_t^0, c_t^i\Big)\Big\}dt \\
&\qquad +\sigma^0(t, c_t^0, c_t^i)dW_t^0 + \sigma(t, c_t^0, c_t^i)dW_t^i, \\
dY_t^i &= -\partial_x \overline{f}\Big(t, X_t^i, -\frac{1}{N}\sum_{j=1}^N Y_t^j, c_t^0, c_t^i\Big)dt + Z_t^{i,0}dW_t^0 + \sum_{j=1}^N Z_t^{i,j}dW_t^j, \qquad (4.3)
\end{aligned}
$$

with

$$
\begin{aligned}
X_0^i &= \xi^i, \\
Y_T^i &= \frac{\delta}{1-\delta}\frac{1}{N}\sum_{j=1}^{N}\partial_x \overline{g}(X_T^j, c_T^0, c_T^j) + \partial_x \overline{g}(X_T^i, c_T^0, c_T^i).
\end{aligned} \tag{4.4}
$$

The existence of a unique solution (4.3) and (4.4) is guaranteed by Theorem 3.3.

With the help of next Lemma, we establish the strong convergence of the finite-agent equilibrium among the homogeneous agents to the MFG limit given in [19]. Let us introduce the following measure arguments based on the solution to (4.1) with (4.2);

$$
\begin{aligned}
\overline{\mu}_t^N &:= \frac{1}{N}\sum_{i=1}^{N}\delta_{\overline{Y}_t^i}, \quad \mathcal{L}_t^0(\overline{Y}_t) := \mathcal{L}(\overline{Y}_t^i|\overline{\mathcal{F}}_t^0), \quad t\in[0,T], \\
\overline{\mu}_g^N &:= \frac{1}{N}\sum_{i=1}^{N}\delta_{\partial_x \overline{g}(\overline{X}_T^i, c_T^0, c_T^i)}, \quad \mathcal{L}_g^0 := \mathcal{L}\big(\partial_x \overline{g}(\overline{X}_T^i, c_T^0, c_T^i)|\overline{\mathcal{F}}_T^0\big).
\end{aligned} \tag{4.5}
$$

Lemma 4.1. *Let Assumption 4.1 be in force. Then we have*

$$
\begin{aligned}
&\lim_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\Big[W_2(\overline{\mu}_t^N, \mathcal{L}_t^0(\overline{Y}_t))^2\Big] = 0, \\
&\lim_{N\to\infty}\mathbb{E}\Big[W_2(\overline{\mu}_g^N, \mathcal{L}_g^0)^2\Big] = 0.
\end{aligned}
$$

Moreover, if there exist some positive constants Γ *and* Γ_g *such that* $\sup_{t\in[0,T]}\mathbb{E}[|\overline{Y}_t^i|^q]^{\frac{1}{q}} \le \Gamma$ *and* $\mathbb{E}\big[|\partial_x \overline{g}(\overline{X}_T^i, c_T^0, c_T^i)|^q\big]^{\frac{1}{q}} \le \Gamma_g$ *for some* $q>4$*, then there exists some constant* C *independent of* N *such that*

$$
\begin{aligned}
&\sup_{t\in[0,T]}\mathbb{E}\Big[W_2(\overline{\mu}_t^N, \mathcal{L}_t^0(\overline{Y}_t))^2\Big] \le C\Gamma^2\epsilon_N, \\
&\mathbb{E}\Big[W_2(\overline{\mu}_g^N, \mathcal{L}_g^0)^2\Big] \le C\Gamma_g^2\epsilon_N,
\end{aligned}
$$

where $\epsilon_N := N^{-2/\max(n,4)}\big(1+\log(N)\mathbf{1}_{N=4}\big)$.

Proof. The claim for $W_2(\overline{\mu}_t^N, \mathcal{L}_t^0(\overline{Y}_t))$ was proved in Theorem 5.1 in [19]. Since $\Big(\partial_x \overline{g}(\overline{X}_T^i, c_T^0, c_T^i)\Big)_{i=1}^N$ are $\overline{\mathcal{F}}_T^0$-conditionally i.i.d. square integrable random variables, the claim for the $W_2(\overline{\mu}_g^N, \mathcal{L}_g^0)$ is established in the same way. Since the time T is fixed, the continuity property used for $W_2(\overline{\mu}_t^N, \mathcal{L}_t^0(\overline{Y}_t))$ is unnecessary. The non-asymptotic estimate on the convergence order in N is the direct consequence of Remark 5.9 in reference [4]. □

The next theorem is the main result of the paper.

Theorem 4.2. *Under Assumption 4.1, let* $(X^i, Y^i, Z^{i,0}, (Z^{i,j})_{j=1}^N)_{i=1}^N$ *and* $(\overline{X}^i, \overline{Y}^i, \overline{Z}^{i,0}, \overline{Z}^{i,i})_{i=1}^N$ *denote the unique strong solution to the* N*-coupled system of FBSDEs* (4.3) *with* (4.4) *and* N*-decoupled FBSDEs of conditional McKean-Vlasov type* (4.1) *with* (4.2)*, respectively. Then,* *there exists some N-independent constant C such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta X_t^i|^2+\sup_{t\in[0,T]}|\Delta Y_t^i|^2+\int_0^T\big(|\Delta Z_t^{i,0}|^2+\sum_{j=1}^N|\Delta Z_t^{i,j}|^2\big)dt\Big]$$
$$\leq C\mathbb{E}\Big[W_2(\overline{\mu}_g^N,\mathcal{L}_g^0)^2+\int_0^T W_2(\overline{\mu}_t^N,\mathcal{L}_t^0(\overline{Y}_t))^2dt\Big], \tag{4.6}$$

where $\Delta X^i:=X^i-\overline{X}^i$, $\Delta Y^i:=Y^i-\overline{Y}^i$, $\Delta Z^{i,0}:=Z^{i,0}-\overline{Z}^{i,0}$ *and* $\Delta Z^{i,j}:=Z^{i,j}-\delta_{i,j}\overline{Z}^{i,i}$.

Proof. Using the notation (3.7), we have for each $1\leq i\leq N$,

$$d\Delta X_t^i=\Big(B(t,Y_t^i,\mu_t^N)-B(t,\overline{Y}_t^i,\mathcal{L}_t^0(\overline{Y}_t))\Big)dt,$$
$$d\Delta Y_t^i=\Big(F(t,X_t^i,\mu_t^N)-F(t,\overline{X}_t^i,\mathcal{L}_t^0(\overline{Y}_t))\Big)dt+\Delta Z_t^{i,0}dW_t^0+\sum_{j=1}^N\Delta Z_t^{i,j}dW_t^j, \tag{4.7}$$

for $t\in[0,T]$ where $\mu_t^N:=\frac{1}{N}\sum_{i=1}^N\delta_{Y_t^i}$ is the empirical measure. To lighten the expression, we omit the arguments (c_t^0,c_t^i), which does not play an important role for the stability analysis below.
First Step: It is important to notice the inequality

$$\Big|\frac{1}{N}\sum_{i=1}^N\overline{Y}_t^i-\mathbb{E}[\overline{Y}_t^i|\overline{\mathcal{F}}_t^0]\Big|\leq W_2\big(\overline{\mu}_t^N,\mathcal{L}_t^0(\overline{Y}_t)\big).$$

This is understood as follows; for an arbitrary pair $\mu,\nu\in\mathcal{P}_2(\mathbb{R}^n)$, we have

$$\Big|\int_{\mathbb{R}^n}x\mu(dx)-\int_{\mathbb{R}^n}y\nu(dy)\Big|=\Big|\int_{\mathbb{R}^n\times\mathbb{R}^n}(x-y)\pi(dx,dy)\Big|\leq\int_{\mathbb{R}^n\times\mathbb{R}^n}|x-y|\pi(dx,dy), \tag{4.8}$$

for any coupling $\pi\in\Pi_2(\mu,\nu)$ with marginals μ and ν. Taking the infimum over $\pi\in\Pi_2(\mu,\nu)$, we get

$$|m(\mu)-m(\nu)|\leq W_1(\mu,\nu)\leq W_2(\mu,\nu),$$

by the definition of the Wasserstein distance (2.1). From Assumption 3.2 (i) and the above observation, one can see that B and F are both Lipschitz continuous in their measure argument with respect to the W_2-distance.

We have

$$\sum_{i=1}^N\langle B(t,Y_t^i,\mu_t^N)-B(t,\overline{Y}_t^i,\mathcal{L}_t^0(\overline{Y}_t)),\Delta Y_t^i\rangle$$
$$=\sum_{i=1}^N\langle B(t,Y_t^i,\mu_t^N)-B(t,\overline{Y}_t^i,\overline{\mu}_t^N),\Delta Y_t^i\rangle+\sum_{i=1}^N\langle B(t,\overline{Y}_t^i,\overline{\mu}_t^N)-B(t,\overline{Y}_t^i,\mathcal{L}_t^0(\overline{Y}_t)),\Delta Y_t^i\rangle$$
$$\leq -N\gamma^l\Big|\frac{1}{N}\sum_{i=1}^N\Delta Y_t^i\Big|^2+C\sum_{i=1}^N W_2(\overline{\mu}_t^N,\mathcal{L}_t^0(\overline{Y}_t))|\Delta Y_t^i|,$$

where we have used Assumption 3.2 (ii). Similar analysis with Assumption 3.1 (iv) and Assumption 3.2 (i) yields

$$\begin{aligned}
&\sum_{i=1}^{N}\langle F(t,X_t^i,\mu_t^N)-F(t,\overline{X}_t^i,\mathcal{L}_t^0(\overline{Y}_t)),\Delta X_t^i\rangle\\
&\leq -\Big(\gamma^f-\frac{L_\varpi^2}{4\gamma^l}\Big)\sum_{i=1}^{N}|\Delta X_t^i|^2+N\gamma^l\Big|\frac{1}{N}\sum_{i=1}^{N}\Delta Y_t^i\Big|^2+C\sum_{i=1}^{N}W_2(\overline{\mu}_t^N,\mathcal{L}_t^0(\overline{Y}_t))|\Delta X_t^i|.
\end{aligned}$$

Since $\Delta X_0^i=0$ for every i, by simple application of Itô-formula and the above estimates, we obtain

$$\begin{aligned}
\sum_{i=1}^{N}\mathbb{E}\Big[\big\langle\Delta X_T^i,\Delta Y_T^i\big\rangle\Big] \quad\leq\quad & -\Big(\gamma^f-\frac{L_\varpi^2}{4\gamma^l}\Big)\sum_{i=1}^{N}\mathbb{E}\int_0^T|\Delta X_t^i|^2dt\\
&+C\sum_{i=1}^{N}\mathbb{E}\int_0^T W_2(\overline{\mu}_t^N,\mathcal{L}_t^0(\overline{Y}_t))\big(|\Delta X_t^i|+|\Delta Y_t^i|\big)dt\ . \qquad (4.9)
\end{aligned}$$

On the other hand, with $\mu_g^N:=\frac{1}{N}\sum_{i=1}^N\partial_x\overline{g}(X_T^i,c_T^0,c_T^i)$, we have from the terminal condition

$$\begin{aligned}
&\sum_{i=1}^{N}\langle G(\mu_g^N,X_T^i)-G(\mathcal{L}_g^0,\overline{X}_T^i),\Delta X_T^i\rangle\\
&=\sum_{i=1}^{N}\langle G(\mu_g^N,X_T^i)-G(\overline{\mu}_g^N,\overline{X}_T^i),\Delta X_T^i\rangle+\sum_{i=1}^{N}\langle G(\overline{\mu}_g^N,\overline{X}_T^i)-G(\mathcal{L}_g^0,\overline{X}_T^i),\Delta X_T^i\rangle,
\end{aligned}$$

where we have omitted (c_t^0,c_T^i) to lighten the notation. Using Assumption 3.1 (iv) and Assumption 3.2 (iii), we have

$$\sum_{i=1}^{N}\mathbb{E}\Big[\big\langle\Delta Y_T^i,\Delta X_T^i\big\rangle\Big]\geq\gamma\sum_{i=1}^{N}\mathbb{E}\big[|\Delta X_T^i|^2\big]-\frac{\delta}{1-\delta}\sum_{i=1}^{N}\mathbb{E}\Big[W_2(\overline{\mu}_g^N,\mathcal{L}_g^0)|\Delta X_T^i|\Big]. \quad (4.10)$$

Combining the two estimates (4.9) and (4.10) gives

$$\begin{aligned}
\sum_{i=1}^{N}\mathbb{E}\Big[|\Delta X_T^i|^2+\int_0^T|\Delta X_t^i|^2dt\Big] \quad\leq\quad & C\sum_{i=1}^{N}\mathbb{E}\int_0^T W_2(\overline{\mu}_t^N,\mathcal{L}_t^0(\overline{Y}_t))\big[|\Delta X_t^i|+|\Delta Y_t^i|\big]dt\\
&+\quad C\mathbb{E}\Big[W_2(\overline{\mu}_g^N,\mathcal{L}_g^0)|\Delta X_T^i|\Big].
\end{aligned}$$

Using Young's inequality and the fact that the random variables such as $\Delta X^i,\Delta Y^i$ have the same distribution for every $1\leq i\leq N$ due to the common coefficient functions, the assumptions on ξ^i and c^i and the structure of probability space, we obtain

$$\begin{aligned}
\mathbb{E}\Big[|\Delta X_T^i|^2+\int_0^T|\Delta X_t^i|^2dt\Big] \quad\leq\quad & C\mathbb{E}\int_0^T\big[W_2(\overline{\mu}_t^N,\mathcal{L}_t^0(\overline{Y}_t))^2+W_2(\overline{\mu}_t^N,\mathcal{L}_t^0(\overline{Y}_t))|\Delta Y_t^i|\big]dt\\
&+\quad C\mathbb{E}\Big[W_2(\overline{\mu}_g^N,\mathcal{L}_g^0)^2\Big], \qquad (4.11)
\end{aligned}$$

for every $1 \le i \le N$.
Second Step: A simple application of Itô-formula to $|\Delta Y_t^i|^2$ gives, for any $t \in [0,T]$,

$$
\begin{aligned}
&\mathbb{E}\Big[|\Delta Y_t^i|^2 + \int_t^T \big(|\Delta Z_s^{i,0}|^2 + \sum_{j=1}^N |\Delta Z_s^{i,j}|^2\big) ds\Big] \\
&\quad = \mathbb{E}\Big[|\Delta Y_T^i|^2 - 2\int_t^T \langle F(s, X_s^i, \mu_s^N) - F(s, \overline{X}_s^i, \mathcal{L}_s^0(\overline{Y}_s)), \Delta Y_s^i\rangle ds\Big] \,. \qquad (4.12)
\end{aligned}
$$

Note that, from Assumption 3.1 (iii) and the estimate (4.8),

$$
\begin{aligned}
|\Delta Y_T^i| &\le |G(\mu_g^N, X_T^i) - G(\overline{\mu}_g^N, \overline{X}_T^i)| + |G(\overline{\mu}_g^N, \overline{X}_T^i) - G(\mathcal{L}_g^0, \overline{X}_T^i)| \\
&\le C\Big(\frac{1}{N}\sum_{j=1}^N |\Delta X_T^j| + |\Delta X_T^i| + W_2(\overline{\mu}_g^N, \mathcal{L}_g^0)\Big).
\end{aligned}
$$

Using this estimate and the exchangeability of variables, we obtain from (4.12) that

$$
\begin{aligned}
&\mathbb{E}\Big[|\Delta Y_t^i|^2 + \int_t^T \big(|\Delta Z_s^{i,0}|^2 + \sum_{j=1}^N |\Delta Z_s^{i,j}|^2\big) ds\Big] \\
&\le C\mathbb{E}\Big[|\Delta X_T^i|^2 + W_2(\overline{\mu}_g^N, \mathcal{L}_g^0)^2\Big] + C\mathbb{E}\int_t^T \big[|\Delta X_s^i| + W_2(\mu_s^N, \mathcal{L}_s^0(\overline{Y}_s))\big]|\Delta Y_s^i| ds \\
&\le C\mathbb{E}\Big[|\Delta X_T^i|^2 + W_2(\overline{\mu}_g^N, \mathcal{L}_g^0)^2\Big] + C\mathbb{E}\int_0^T \big(|\Delta X_s^i|^2 + W_2(\overline{\mu}_s^N, \mathcal{L}_s^0(\overline{Y}_s))^2\big) ds + C\mathbb{E}\int_t^T |\Delta Y_s^i|^2 ds.
\end{aligned}
$$

Here, we have used the triangle inequality and the fact that

$$
\mathbb{E}\Big[W_2(\mu_s^N, \overline{\mu}_s^N)^2\Big] \le \mathbb{E}\Big[\frac{1}{N}\sum_{j=1}^N |Y_s^j - \overline{Y}_s^j|^2\Big] = \mathbb{E}|\Delta Y_s^i|^2. \qquad (4.13)
$$

By applying the backward Gronwall's inequality and the estimate (4.11), we get

$$
\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\Big[|\Delta Y_t^i|^2\Big] + \mathbb{E}\int_0^T \big(|\Delta Z_t^{i,0}|^2 + \sum_{j=1}^N |\Delta Z_t^{i,j}|^2\big) dt \\
&\le C\mathbb{E}\Big[W_2(\overline{\mu}_g^N, \mathcal{L}_g^0)^2 + \int_0^T \big(W_2(\overline{\mu}_t^N, \mathcal{L}_t^0(\overline{Y}_t))^2 + W_2(\overline{\mu}_t^N, \mathcal{L}_t^0(\overline{Y}_t))|\Delta Y_t^i|\big) dt\Big].
\end{aligned}
$$

Using Young's inequality, we obtain

$$
\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\Big[|\Delta Y_t^i|^2\Big] + \mathbb{E}\int_0^T \big(|\Delta Z_t^{i,0}|^2 + \sum_{j=1}^N |\Delta Z_t^{i,j}|^2\big) dt \\
&\qquad \le C\mathbb{E}\Big[W_2(\overline{\mu}_g^N, \mathcal{L}_g^0)^2 + \int_0^T W_2(\overline{\mu}_t^N, \mathcal{L}_t^0(\overline{Y}_t))^2 dt\Big], \qquad (4.14)
\end{aligned}
$$

from which and (4.11), we also have

$$\mathbb{E}\Big[|\Delta X_T^i|^2+\int_0^T|\Delta X_t^i|^2dt\Big]\le C\mathbb{E}\Big[W_2(\overline{\mu}_g^N,\mathcal{L}_g^0)^2+\int_0^T W_2(\overline{\mu}_t^N,\mathcal{L}_t^0(\overline{Y}_t))^2dt\Big]. \tag{4.15}$$

The inequality (4.6) now easily follows from (4.14), (4.15) and the standard application of the Burkholder-Davis-Gundy inequality. □

Under the conditions used in Theorem 4.2, the market clearing price for the homogeneous agents is given by

$$\varpi_t^{\mathrm{Ho}}:=-\frac{1}{N}\sum_{i=1}^N Y_t^i,\quad t\in[0,T],$$

where $(Y^i)_{i=1}^N$ is the solution to the N-coupled system of FBSDEs (4.3) with (4.4). On the other hand, the price process in the mean-field limit is given by

$$\varpi_t^{\mathrm{MFG}}:=-\mathbb{E}\Big[\overline{Y}_t^i|\overline{\mathcal{F}}_t^0\Big],\quad t\in[0,T], \tag{4.16}$$

which is proven to clear the market asymptotically in the large population limit [19].

Corollary 4.1. *Let Assumption 4.1 be in force. With the above notations, we have*

$$\begin{aligned}\sup_{t\in[0,T]}\mathbb{E}\Big[|\varpi_t^{\mathrm{Ho}}-\varpi_t^{\mathrm{MFG}}|^2\Big]+\mathbb{E}\Big[\sup_{t\in[0,T]}\big|\mathbb{E}[\varpi_t^{\mathrm{Ho}}|\overline{\mathcal{F}}_t^0]-\varpi_t^{\mathrm{MFG}}\big|^2\Big]&\\ \le C\Big(\sup_{t\in[0,T]}\mathbb{E}\Big[W_2(\overline{\mu}_t^N,\mathcal{L}_t^0(\overline{Y}_t))^2\Big]+\mathbb{E}\Big[W_2(\overline{\mu}_g^N,\mathcal{L}_g^0)^2\Big]\Big)&\ ,\end{aligned}$$

where C is some N-independent constant.

Proof. Using (4.8), we have

$$\begin{aligned}\big|\varpi_t^{\mathrm{Ho}}-\varpi_t^{\mathrm{MFG}}\big|^2&=\big|m(\mu_t^N)-m(\mathcal{L}_t^0(\overline{Y}_t))\big|^2\\ &\le W_2(\mu_t^N,\mathcal{L}_t^0(\overline{Y}_t))^2\le 2W_2(\mu_t^N,\overline{\mu}_t^N)^2+2W_2(\overline{\mu}_t^N,\mathcal{L}_t^0(\overline{Y}_t))^2.\end{aligned}$$

The desired estimate for the first term now follows from (4.13). For the second term,

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|\mathbb{E}[\varpi_t^{\mathrm{Ho}}|\overline{\mathcal{F}}_t^0]-\varpi_t^{\mathrm{MFG}}\big|^2\Big]=\mathbb{E}\Big[\sup_{t\in[0,T]}\Big|\frac{1}{N}\sum_{i=1}^N\mathbb{E}\big[Y_t^i-\overline{Y}_t^i|\overline{\mathcal{F}}_t^0\big]\Big|^2\Big]\le\frac{1}{N}\sum_{i=1}^N\mathbb{E}\Big[\sup_{t\in[0,T]}|Y_t^i-\overline{Y}_t^i|^2\Big],$$

and hence the desired estimate immediately follows. □

Stability of the market price for the heterogeneous agents

Suppose that the N agents have the common discount parameter δ and the common rate of the trading fee Λ to be paid to the securities exchange. Instead of the homogeneous agents, we now consider the case where the agents have different cost functions and different order-flow from their clients; $(l_i,\sigma_i^0,\sigma_i,\overline{f}_i,\overline{g}_i), 1\le i\le N$. It is interesting to study the condition under which the heterogeneous market converges to the MFG limit. From a practical perspective, it is also important to know how accurately we can approximate the market price by $\varpi^{\rm MFG}$.

Proposition 4.1. *Assume that the coefficients* (δ,Λ) *and* $(l_i,\sigma_i^0,\sigma_i,\overline{f}_i,\overline{g}_i)_{i=1}^N$ *satisfy the conditions in Assumptions 3.1 and 3.2, and that* $(l,\sigma^0,\sigma,\overline{f},\overline{g})$ *satisfy the conditions in Assumption 4.1. In addition to the notations used in this section, let us denote the solution to* (3.5) *with* (3.6) *by* $(\check{X}^i,\check{Y}^i,\check{Z}^{i,0},(\check{Z}^{i,j})_{j=1}^N)_{i=1}^N$. *Then there exists some* N *independent constant* C *such that*

$$\begin{aligned}&\sum_{i=1}^N\mathbb{E}\Big[\sup_{t\in[0,T]}|\Delta\check{X}_t^i|^2+\sup_{t\in[0,T]}|\Delta\check{Y}_t^i|^2+\int_0^T\big(|\Delta\check{Z}_t^{i,0}|^2+\sum_{j=1}^N|\Delta\check{Z}_t^{i,j}|^2\big)dt\Big]\\&\quad\le CN\mathbb{E}\Big[W_2(\overline{\mu}_g^N,\mathcal{L}_g^0)^2+\int_0^T W_2(\overline{\mu}_t^N,\mathcal{L}_t^0(\overline{Y}_t))^2dt\Big]\\&\quad+C\sum_{i=1}^N\mathbb{E}\Big[|\overline{G}_i|^2+\int_0^T\big(|\overline{F}_i(t)|^2+|\overline{B}_i(t)|^2+|\overline{\sigma}_i^0(t)|^2+|\overline{\sigma}_i(t)|^2\big)dt\Big]\ ,\end{aligned}$$

where $\Delta\check{X}^i:=\check{X}^i-\overline{X}^i$, $\Delta\check{Y}^i:=\check{Y}^i-\overline{Y}^i$, $\Delta\check{Z}^{i,0}:=\check{Z}^{i,0}-\overline{Z}^{i,0}$, $\Delta\check{Z}^{i,j}=\check{Z}^{i,j}-\delta_{i,j}\overline{Z}^{i,i}$ *and*

$$\begin{aligned}&\overline{B}_i(t):=(l_i-l)\Big(t,Y_t^i,\varpi_t^{\rm Ho},c_t^0,c_t^i\Big),\qquad \overline{F}_i(t):=-\big(\partial_x\overline{f}_i-\partial_x\overline{f}\big)\Big(t,X_t^i,\varpi_t^{\rm Ho},c_t^0,c_t^i\Big),\\&\overline{G}_i:=\frac{\delta}{1-\delta}\sum_{j=1}^N\big(\partial_x\overline{g}_j-\partial_x\overline{g}\big)(X_T^j,c_T^0,c_T^j)+(\partial_x\overline{g}_i-\partial_x\overline{g})(X_T^i,c_T^0,c_T^i).\end{aligned}$$

Proof. This is the direct consequence of Proposition 3.1 and Theorem 4.2. □

From Theorem 3.2 and Theorem 3.3, we know that the market clearing price among the N heterogeneous agents is given by

$$\varpi_t^{\rm He}:=-\frac{1}{N}\sum_{i=1}^N\check{Y}_t^i,\quad t\in[0,T].$$

The next corollary gives the stability result of the market price around the mean-field limit.

Theorem 4.3. *Under the assumptions used in Proposition 4.1, there exists some* N *independent constant* C *such that*

$$\begin{aligned}&\sup_{t\in[0,T]}\mathbb{E}\Big[|\varpi_t^{\rm He}-\varpi_t^{\rm MFG}|^2\Big]+\mathbb{E}\Big[\sup_{t\in[0,T]}\big|\mathbb{E}[\varpi_t^{\rm He}|\overline{\mathcal{F}}_t^0]-\varpi_t^{\rm MFG}\big|^2\Big]\\&\le C\Big(\sup_{t\in[0,T]}\mathbb{E}\Big[W_2(\overline{\mu}_t^N,\mathcal{L}_t^0(\overline{Y}_t))^2\Big]+\mathbb{E}\Big[W_2(\overline{\mu}_g^N,\mathcal{L}_g^0)^2\Big]\Big)\\&\qquad+C\frac{1}{N}\sum_{i=1}^N\mathbb{E}\Big[|\overline{G}_i|^2+\int_0^T\big(|\overline{F}_i(t)|^2+|\overline{B}_i(t)|^2+|\overline{\sigma}_i^0(t)|^2+|\overline{\sigma}_i(t)|^2\big)dt\Big]\ .\end{aligned}$$

Proof. The desired estimate follows from Proposition 4.1. It is easy to check

$$\begin{aligned}|\varpi_t^{\mathrm{He}} - \varpi_t^{\mathrm{MFG}}|^2 &\leq 2\Big|\frac{1}{N}\sum_{i=1}^{N}(\check{Y}_t^i - \overline{Y}_t^i)\Big|^2 + 2|m(\overline{\mu}_t^N) - \varpi_t^{\mathrm{MFG}}|^2 \\ &\leq 2\frac{1}{N}\sum_{i=1}^{N}|\Delta\check{Y}_t^i|^2 + 2W_2(\overline{\mu}_t^N, \mathcal{L}_t^0(\overline{Y}))^2,\end{aligned}$$

which gives the estimate for the first term. The estimate for the second term is the immediate consequence of the inequality

$$\mathbb{E}\Big[\sup_{t\in[0,T]}\big|\mathbb{E}[\varpi_t^{\mathrm{He}}|\overline{\mathcal{F}}_t^0] - \varpi_t^{\mathrm{MFG}}\big|^2\Big] \leq \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\Big[\sup_{t\in[0,T]}|\check{Y}_t^i - \overline{Y}_t^i|^2\Big].$$

□

Remark 4.2. *Theorem 4.3 implies that the market clearing price converges to the mean-field limit* ϖ^{MFG} *if the difference of coefficients functions* $(\overline{G}_i, \overline{F}_i, \overline{B}_i, \overline{\sigma}_i^0, \overline{\sigma}_i)_{i\geq 1}$ *converges to zero in* $\mathbb{L}^2(\mathcal{F}_T;\mathbb{R})$ *and* $\mathbb{H}^2(\mathbb{F};\mathbb{R})$ *in the large population limit* $N\to\infty$*. It is clear that any deviation from the limit coefficient functions* $(\overline{g}, \overline{f}, l, \sigma^0, \sigma)$ *among the finite number of agents does not affect this convergence.*

5 Conclusions and Discussions

In this work, we prove the existence of a unique market clearing equilibrium among the heterogeneous agents of finite population size under the assumption that they are the price takers. We show the strong convergence to the corresponding mean-field limit given in [19] under appropriate conditions. In particular, we provide the stability relation between the market clearing price for the heterogeneous agents and that for the homogeneous mean-field limit. An extension to multiple populations [17] as studied in Section 6 of [19] looks straightforward. One of the important topics of the future research is to allow the presence of a major player in the securities market. If we adopt the concept of the Stackelberg equilibrium with a leader (major agent) and followers (minor agents), careful investigation of the coupled system of FBSDEs with appropriate convexity and monotone assumptions may prove the existence of the market clearing equilibrium. It may be also possible to analyze the mean field limit of the minor agents in the presence of one major agent.

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