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Timing Games with Irrational Types: Leverage-Driven Bubbles and Crash-Contingent Claims

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Abstract

This study investigates a timing game with irrational types; each player selects a time in a fixed time interval, and the player who selects the earliest time wins the game. We assume the possibility of irrational types in that each player is irrational with a positive probability, thus selecting the terminal time. We show that there exists the unique Nash equilibrium; according to it, every player never selects the initial time.

As an application, we analyze a strategic aspect of leverage-driven bubbles; even if a company is unproductive, its stock price grows up according to an exogenous reinforcement pattern. During the bubble, this company is willing to raise huge funds by issuing new shares. We regard players as arbitrageurs, who decide whether to ride the bubble or burst it. We demonstrate two models, which are distinguished by whether crash-contingent claim, i.e., contractual agreement such that the purchaser of this claim receives a promised monetary amount from its seller if and only if the bubble crashes, is available.

The availability of this claim deters the bubble; without crash-contingent claim, the bubble emerges and persists even if the degree of reinforcement is insufficient. Without crash-contingent claim, high leverage ratio fosters the bubble, while with crash-contingent claim, it rather deters the bubble.

Keywords: Timing Games with Irrational Types, Uniqueness, Awareness Heterogeneity, Leverage-Driven Bubbles, Crash-Contingent Claim.
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1. Introduction

This study investigates a *timing game with irrational types*; each player selects a time to quit the game in a fixed time interval $[0,1]$, and the player who selects the earliest time among all players wins the game. The winner's payoff is increasing with respect to this time. Provided all players are rational, it is the only Nash equilibrium in this game for each player to quit immediately at the initial time; rational players miss the opportunity to obtain greater future rewords because of their excess tail-chasing competition.

This study assumes the possibility of irrational types in that each player is *irrational* with a positive probability, thus naively selecting the terminal time. With the possibility of such irrational types, it is no longer a Nash equilibrium for all rational players to quit immediately. Any rational player is willing to postpone the time, because he can win against irrational players, receiving a larger wining payoff.

This study theoretically shows that there exists the unique (mixed strategy) Nash equilibrium in the timing game with irrational types. According to this equilibrium, every player never quits at the initial time.

By applying this theoretical finding, we analyze a strategic aspect of *bubbles and crashes* in a stock market. There is a company that has no profitable business opportunity. This company's stock price nevertheless grows up as a phenomenon of bubble according to an exogenously given reinforcement pattern. During the bubble, the company is willing to raise funds by issuing new shares for its owner/manager's personal usage, which is unproductive and even socially wasteful. This study investigates the possibility that such a harmful bubble can emerge and persist long.

There are multiple arbitrageurs each of whom decides when to sell out his (or her) shareholding and burst the bubble. Before selling out, he (or she) has to ride the bubble by continuing to purchase newly issued shares. Since these arbitrageurs have budgetary constraints, they borrow money from noise traders who have a plenty of money but are naïve in that they misperceive the company's fundamental value, reinforce their misperception, and are even unaware of the crash risk and their misperception. In contrast to these noise traders, rational arbitrageurs are well aware of them.

By treating such noise traders as a reduced form, this study demonstrates two models of bubbles and crashes as specifications of timing games with irrational types. These models are distinguished by whether *crash-contingent claims* are available; we define crash-contingent claim as the contractual agreement such that the purchaser of this claim receives a promised monetary amount from its seller if and only if the bubble crashes. Because of the above-mentioned awareness heterogeneity, arbitrageurs can purchase crash-contingent claims from noise traders on a quite favorable condition, i.e., for nothing.

We can clarify a significant difference between these models with respect to the impact of leverage-ratio regulation. Without crash-contingent claim, the permission of arbitrageurs' high leverage ratio fosters a leverage-driven bubble to emerge and persist long, while with the availability of crash-contingent claim, it rather deters the bubble. We further argue that to deter the bubble, it would be an effective policy method that we will make crash-contingent claims available, and then make the leverage ratio cap as weak as possible. Note that without crash-contingent claim, the leverage-driven bubble emerges and persists even if the degree of noise traders' reinforcement is insufficient.

It is crucial to assume that the purchaser of crash-contingent claim can receive the promised payment from its seller even if this purchaser successfully sells out his shareholding before the crash. It is also crucial to assume that he (or she) is not exempted from his debt obligation even if he fails to sell out before the crash. These assumptions imply that the purchase of crash-contingent claim increases the difference between the winner's payoff and the loser's payoff, thus urging the purchasers to sell out his shareholding at early times.

Matsushima (2013) formulated timing games with irrational types by assuming that the winner's payoff has exponential growth and that the loser's payoff is constant across time. This study generalizes this formulation by eliminating these assumptions. This generalization is necessary for specifying the two models in this study. Matsushima (2013) applied timing games with irrational types to an issue of bubbles and crashes, but did not consider the company's fund-raising and the arbitrageurs' borrowing activity; the bubbles that Matsushima (2013) investigated are not leverage-driven and not socially harmful at all. In contrast, this study seriously takes such fund-raising and leverage into account.

Crash-contingent claim could be regarded as a version of *naked credit default swap*; the purchaser can receive the promised monetary payment from its seller irrespective of whether he holds the underlying assets. In this respect, Che and Sethi (2010), Geanakoplos (2010), and Fostel and Geanakoplos (2012) are related to this study. These works commonly assumed prior heterogeneity, while this study assumes awareness heterogeneity that originates in Abreu and Brunnermeier (2003).

Abreu and Brunnermeier (2003) formulated the stock market as a timing game similar to this study. However, Abreu and Brunnermeier (2003) assumed an aspect of informational asymmetry termed sequential awareness, while this study instead assumes the possibility of arbitrageurs' irrationality.[1]

We should distinguish crash-contingent claim from put option, a popular bailout policy method; it gives its purchaser the right to sell the asset at a promised price if the bubble crashes. The purchaser of put option must keep holding the underlying asset to receive the promised price. Because of this, the availability of put option generally facilitates bubbles.

We should also distinguish crash-contingent claim from covered credit default swap; the payment of a covered credit default swap is utilized only for paying off its purchaser's debt obligation. In contrast to crash-contingent claim or naked credit default swap, the availability of covered credit default swap bears no influence on arbitrageurs' incentive.

The organization of this study is as follows. Section 2 defines timing games with irrational types and shows characterization theorems concerning the existence and uniqueness of Nash equilibrium. Section 3 introduces a stock-market model to explain leverage-driven bubbles without crash-contingent claims, and shows that the permission of high leverage ratio motivates arbitrageurs to ride the bubble even if the degree of noise traders' reinforcement is insufficient. Section 4 shows that the introduction of crash-contingent claim rather discourages arbitrageurs to ride the bubble, and that with the availability of crash-contingent claims, the permission of high leverage ratio discourages arbitrageurs to ride the bubble. Section 5 concludes.

[1] See Brunnermeier and Oehmke (2013) for a survey on bubbles, crashes, and crises. See also the limit-to-arbitrage literature such as De Long et al. (1990) and Shleifer and Vishny (1992).

2. Timing Games with Irrational Types

We define a *timing game with irrational types* as follows.[2] Consider a finite set of players $N=\{1,2,...,n\}$, where $n\geq 2$. Let $A_i=[0,1]$ denote the set of all pure strategies for each player $i\in N$. A mixed strategy, shortly a *strategy*, for player i is defined as a cumulative distribution $q_i:A_i\rightarrow[0,1]$; $q_i(a_i)$ denotes the probability that player i selects a pure strategy that is equal to or less than $a_i\in[0,1]$, which is non-decreasing and right-continuous. We assume that there exists a pure strategy $a_i\in[0,1)$ such that $q_i(a_i)>0$. We will simply write $q_i=a_i$ when player i selects a_i with certainty. Let Q_i denote the set of all strategies for player i. Let $Q=\underset{i\in N}{\times}Q_i$ and $q=(q_i)_{i\in N}\in Q$. A strategy profile $q\in Q$ is said to be *symmetric* if $q_i=q_1$ for all $i\in N$.

Fix an arbitrary real number $\varepsilon\in(0,1)$. We assume that each player is rational with a probability $1-\varepsilon>0$, while he (or she) is *irrational* with the remaining probability $\varepsilon>0$. If player i is rational, he plays the game according to his selected strategy q_i. If he is irrational, he selects $a_i=1$ with certainty. Whether each player is rational or irrational is determined independently with each other, and is unknown to the other players.

Consider an arbitrary pure strategy profile $a=(a_i)_{i\in N}\in A\equiv\underset{i\in N}{\times}A_i$ and an arbitrary nonempty subset of players $H\subset N$. Suppose that any player in H is rational, while any player in $N\setminus H$ is irrational. Let

$$\tau=\tau(a,H)\equiv\min_{j\in H}a_j \quad\text{and}\quad l=l(a,H)\equiv\left|\{j\in H\mid a_j=\tau(a,H)\}\right|.$$

With a probability $1/l$, each player $i\in H$ who selects the minimum among all players τ wins the game and earns the winner's payoff $\bar{v}(\tau)$. With the remaining

[2] This a generalization of Matsushima (2013), which is closely related to the reputation theory such as Kreps et al. (1982).

probability $\frac{l-1}{l}$, he loses the game and earns the loser's payoff $\underline{v}(\tau)$. Any player who selects a pure strategy that is larger than τ loses the game. We assume that

$$\overline{v}(\tau) > \underline{v}(\tau) \quad \text{for all} \quad \tau \in [0,1],$$

both $\overline{v}(\tau)$ and $\underline{v}(\tau)$ are differentiable, and $\overline{v}(\tau)$ is non-decreasing;

$$\overline{v}'(\tau) \equiv \frac{\partial \overline{v}(\tau)}{\partial \tau} \geq 0 \quad \text{for all} \quad \tau \in [0,1].$$

The expected payoff for a rational player $i \in H$ is given by

$$v_i(H,a) = \frac{1}{l}\overline{v}(\tau) + \frac{l-1}{l}\underline{v}(\tau) \qquad \text{if} \quad a_i = \tau,$$

and

$$v(H,a) \equiv \underline{v}(\tau) \qquad \text{if} \quad a_i > \tau.$$

We define the *payoff function* $u_i(\cdot,\varepsilon): Q \to R$ for each player $i \in N$ as the expected value of $v_i(H,a)$;

(1) $$u_i(q,\varepsilon) \equiv E[v_i(H,a) \mid q, \varepsilon, i \in H],$$

where $E[\cdot \mid q, \varepsilon, i \in H]$ denotes the expectation operator provided that player i is rational. A strategy profile $q \in Q$ is said to be a *Nash equilibrium* if

$$u_i(q,\varepsilon) \geq u_i(q_i', q_{-i}, \varepsilon) \quad \text{for all} \quad i \in N \quad \text{and all} \quad q_i' \in Q_i.$$

An interpretation of timing game with irrational types is as follows. Each player selects a time to quit the game during the time interval $[0,1]$. A player, who selects the earliest time among all players, wins the game. The earlier this time is, the lesser the winner's payoff is. Without irrational types ($\varepsilon = 0$), it is the only Nash equilibrium for all players to quit the game immediately at the initial time (time zero). With irrational types ($\varepsilon > 0$), however, a rational player may have incentive to postpone the time from time zero to improve the winning payoff; he can still win the game against irrational players.

Timing games with irrational types have the potential ability to analyze various economic problems. An example is the common resource problem; there is a common resource (tree), which grows over time but rots at the terminal time for some exogenous reason. There are multiple agents who can cut down this resource at any time and consume all of its fruit without sharing them with the other agents. Each agent expects for any other agent to find it costly to cut down the resource, i.e., he is irrational as per this paper's terminology, with a probability $\varepsilon > 0$.

This study intensively analyzes a strategic aspect of bubbles and crashes in a stock market as an application of timing games with irrational types. We regard a player as an *arbitrageur*. The bubble grows up during a time interval $[0,1]$, but it automatically crashes at the terminal time 1. Even before the terminal time, the bubble crashes if an arbitrageur sells out his (or her) shareholding. Any strategic arbitrageur attempts to sell out earlier than the other arbitrageurs; otherwise, he fails to sell out before the crash, earning nothing. The detail of bubbles and crashes will be explained in the later sections.

The probability that the timing game ends at or before time $t \in [0,1]$ is given by

$$D(t;q,\varepsilon) \equiv 1 - \prod_{i \in N} \{1-(1-\varepsilon)q_i(t)\}.$$

We define the *hazard rate* by

$$\theta(t) \equiv \frac{D'(t;q,\varepsilon)}{1-D(t;q,\varepsilon)},$$

where $D'(t:q,\varepsilon) \equiv \dfrac{\partial D(t:q,\varepsilon)}{\partial t}$. If $q \in Q$ is symmetric, we can rewrite this hazard rate as

$$\theta(t) = \frac{n(1-\varepsilon)q_1{}'(t)}{1-(1-\varepsilon)q_1(t)}.$$

The probability that the timing game ends at or before time t, provided player $i \in N$ has never quitted before, is given by

$$D_i(t;q_{-i},\varepsilon) = 1 - \prod_{j \in N\setminus\{i\}} \{1-(1-\varepsilon)q_j(t)\}.$$

We can rewrite the payoff (1) as

$$u_i(t,q_{-i},\varepsilon) = \int_{\tau=0}^{t} \underline{v}(\tau)dD_i(\tau:q_{-i},\varepsilon) + \overline{v}(t)\{1-D_i(t:q_{-i},\varepsilon)\}.$$

Hence, we can derive the *first-order condition* for Nash equilibrium as

(2) $$\{\overline{v}(t)-\underline{v}(t)\}D_i'(t:q_{-i},\varepsilon) = \overline{v}'(t)\{1-D_i(t:q_{-i},\varepsilon)\},$$

where $D_i'(t:q_{-i},\varepsilon) \equiv \dfrac{\partial D_i(t:q_{-i},\varepsilon)}{\partial t}$. We define the *relative future benefit* at time $t \in [0,1]$ as

$$R(t) \equiv \frac{\overline{v}'(t)}{\overline{v}(t) - \underline{v}(t)}.$$

Whenever the first-order condition holds, the hazard rate $\theta(t)$ is proportional to the relative future benefit;

$$\theta(t) = \frac{n}{n-1} R(t).$$

We specify a symmetric strategy profile $\tilde{q} = \tilde{q}(\varepsilon) \in Q$ as follows:

(3) $$\tilde{q}_1(t) = \frac{1 - \{1 - (1-\varepsilon)\tilde{q}_1(\tilde{\tau})\} \exp[-\frac{1}{n-1}\int_{\tau=\tilde{\tau}}^{t} R(\tau) d\tau]}{1-\varepsilon} \quad \text{for all } t \in [\tilde{\tau}, 1],$$

and

$$\tilde{q}_1(t) = 0 \quad \text{for all } t \in [0, \tilde{\tau}).$$

where we define the *critical time* $\tilde{\tau} = \tilde{\tau}(\varepsilon) \in [0,1)$ in ways that either

(4) $$\tilde{\tau} \geq 0 \text{ and } \varepsilon = \exp[-\frac{1}{n-1}\int_{\tau=\tilde{\tau}}^{1} R(\tau) d\tau],$$

or

$$\tilde{\tau} = 0 \text{ and } \varepsilon = \{1 - (1-\varepsilon)\tilde{q}_1(0)\} \exp[-\frac{1}{n-1}\int_{\tau=0}^{1} R(\tau) d\tau].$$

According to $\tilde{q}$, no player quits the game before the critical time $\tilde{\tau}$. After the critical time $\tilde{\tau}$, the timing game ends stochastically according to the hazard rate $\theta(t) = \dfrac{n}{n-1} R(t)$.

From (4), the greater the relative future benefit $R(t)$ is, the later the critical time $\tilde{\tau}$ is. The greater the relative future benefit $R(t)$ is, the smaller the probability $\tilde{q}_1(t)$ is. Hence, the arbitrageurs tend to quit later, i.e., the timing game persists longer as the relative future benefits are greater.

The following theorem shows a necessary and sufficient condition for the specified symmetric strategy profile $\tilde{q}$ to be a Nash equilibrium. It also shows a (almost) necessary and sufficient condition for $\tilde{q}$ to be the *unique* Nash equilibrium. This theorem is a generalization of Matsushima (2013), which assumed that the winner's payoff has exponential growth and the loser's payoff is constant across time.

Theorem 1: *The symmetric strategy profile* $\tilde{q}$ *is a Nash equilibrium if and only if*

$$\exp[-\frac{1}{n-1}\int_{\tau=0}^{1} R(\tau)d\tau] \leq \varepsilon .$$

It is a unique Nash equilibrium if the strict inequality holds;

$$\exp[-\frac{1}{n-1}\int_{\tau=0}^{1} R(\tau)d\tau] < \varepsilon .$$

Proof: For every $\hat{\tau} \in [\tilde{\tau},1]$, we specify a symmetric strategy profile $q^{\hat{\tau}} = (q_i^{\hat{\tau}})_{i\in N} \in Q$ as follows:

$$q_1^{\hat{\tau}}(t) = \tilde{q}_1(t) \quad \text{for all} \quad t \in [\hat{\tau},1],$$

and

$$q_1^{\hat{\tau}}(t) = \tilde{q}_1(\hat{\tau}) \quad \text{for all} \quad t \in [0,\hat{\tau}).$$

According to $q^{\hat{\tau}}$, any rational player selects time zero with probability $\tilde{q}_1(0) = \tilde{q}_1(\hat{\tau})$. After time zero, he never quits the game before time $\hat{\tau}$. After time $\hat{\tau}$, he plays the game according to $\tilde{q}_i$.

The Lemma: *A symmetric strategy profile* $q \in Q$ *is a Nash equilibrium if and only if there exists* $\hat{\tau} \in [\tilde{\tau},1]$ *such that*

$$q = q^{\hat{\tau}},$$

(5) $$u_1(0,q_{-1},\varepsilon) = u_1(\hat{\tau},q_{-1},\varepsilon) \quad \textit{whenever} \quad \hat{\tau} < 1 \quad \textit{and} \quad q_1(0) > 0,$$

and

(6) $$u_1(0,q_{-1},\varepsilon) \geq u_1(\hat{\tau},q_{-1},\varepsilon) \quad \textit{whenever} \quad \hat{\tau} = 1.$$

Proof: Consider an arbitrary symmetric Nash equilibrium $q \in Q$. Clearly, (6) is necessary and sufficient for the Nash equilibrium property if $\hat{\tau} = 1$.

Assume $\hat{\tau} < 1$. We show that $q_1(\tau)$ is continuous. Suppose that $q_1(\tau)$ is not continuous; there exists $\tau' > 0$ such that $\lim_{\tau \uparrow \tau'} q_1(\tau) < q_1(\tau')$. Then, by selecting any time slightly earlier than τ', any player can dramatically increase his winning probability. This implies that no player selects τ'. This is a contradiction.

Let

$$\hat{\tau} = \max\{\tau \in (0,1] : q_1(\tau) = q_1(0)\}.$$

We show that $q_1(\tau)$ is increasing in $[\hat{\tau},1]$. Suppose that $q_1(\tau)$ is not increasing in $[\hat{\tau},1]$; there exist $\tau' \in [\hat{\tau},1]$ and $\tau'' \in [\hat{\tau},1]$ such that $\tau' < \tau''$, $q_1(\tau') = q_1(\tau'')$, and the selection of τ' is a best response. Since no player selects any τ in (τ', τ''), it follows from the continuity of q that by selecting τ'' instead of τ', a player can increase his winner's payoff without decreasing his winning probability. This is a contradiction.

Since $q_1(\tau)$ is increasing in $[\hat{\tau},1]$, any selection $\tau \in [\hat{\tau},1]$ must be a best response; the first-order condition must hold for all $\tau \in [\hat{\tau},1]$, implying $q = q^{\hat{\tau}}$. Since $\hat{\tau} < 1$ and $\bar{v}(t)$ is increasing, it follows that $q^{\hat{\tau}}$ is a Nash equilibrium if and only if

$$u_1(0, q_{-1}, \varepsilon) = u_1(\hat{\tau}, q_{-1}, \varepsilon) \quad \text{whenever} \quad q_1(0) > 0.$$

This implies that (5) is necessary and sufficient.

Q.E.D.

We prove the first part of Theorem 1 as follows. Since $\tilde{\tau} < 1$ and $\tilde{q} = q^{\tilde{\tau}}$, it follows from the Lemma that $\tilde{q}$ is a Nash equilibrium if and only if

$$\text{either} \quad \tilde{q}_1(0) = 0 \quad \text{or} \quad u_1(0, \tilde{q}_{-1}, \varepsilon) = u_1(\tilde{\tau}, \tilde{q}_{-1}, \varepsilon).$$

The weak inequality in Theorem 1 implies (5), that is, $\tilde{q}_1(0) = 0$; the weak inequality in Theorem 1 implies that $\tilde{q}$ is a Nash equilibrium.

Suppose that the weak inequality in Theorem 1 does not hold; $\tilde{\tau} = 0$ and $\tilde{q}_1(0) > 0$. This contradicts the Nash equilibrium property; any rational player prefers time zero to any time slightly later than time zero, because he can dramatically increase his winning probability without any substantial decrease in the winner's payoff.

We prove the latter part of Theorem 1 as follows. From the strict inequality in Theorem 1, it follows that the property (4) holds and $\tilde{\tau} > 0$. This along with the Lemma implies that any symmetric Nash equilibrium q must satisfy $q = \tilde{q}$.

Next, we show that $\tilde{q}$ is a unique Nash equilibrium even if we consider all asymmetric strategy profiles. We set any Nash equilibrium $q \in Q$ arbitrarily. First, we show that $q_i(\tau)$ must be continuous. Suppose that $q_i(\tau)$ is not continuous; there exists $\tau' > 0$ such that $\lim_{\tau \uparrow \tau'} q_i(\tau) < q_i(\tau')$. Then, any other player can drastically increase his winning probability by selecting any time slightly earlier than τ'. Hence, no other player selects any time that is either the same as or slightly later than τ'; player i can postpone the time without decreasing his winning probability. This is a contradiction.

Let

$$\tau^1 = \max\{\tau \in (0,1] : q_i(\tau) = q_i(0) \textit{ for all } i \in N\}.$$

Second, we show that $D(\tau;q)$ must be increasing in $[\tau^1,1]$. Suppose that $D(\tau;q)$ is not increasing in $[\tau^1,1]$; there exist $\tau' \in (\tau^1,1]$ and $\tau'' \in (\tau^1,1]$ such that $\tau' < \tau''$, $D(\tau';q) = D(\tau'';q)$, and the selection of τ' is a best response for some player. Since no player selects any time τ in (τ',τ''), it follows from the continuity of q that, by selecting τ'' instead of τ', any player can postpone the time from τ' to τ'' without decreasing his winning probability. This is a contradiction.

Third, we show that q must be symmetric. Suppose that q is asymmetric. The strict inequality in Theorem 1 implies that the selection of time zero is a dominated strategy. Hence, we have $\tau^1 > 0$, and

$$q_i(\tau) = 0 \text{ for all } i \in N \text{ and } \tau \in [0,\tau^1].$$

Since q is continuous and $D(\tau;q)$ is increasing in $[\tau^1,1]$, there exist $\tau' > 0$, $\tau'' > \tau'$, and $i \in N$ such that

$$q_1(t) = q_j(t) \text{ for all } j \in N \text{ and } t \in [0,\tau'],$$

$$\text{(7)} \qquad \frac{\frac{\partial D_i(\tau;q)}{\partial t}}{1 - D_i(\tau;q)} > \min_{h \neq i} \frac{\frac{\partial D_h(\tau;q)}{\partial t}}{1 - D_h(\tau;q)} \text{ for all } t \in (\tau',\tau'')$$

and

$$(8)\qquad \frac{\dfrac{\partial D_i(\tau'';q)}{\partial t}}{1-D_i(\tau'';q)} = \min_{h\neq i} \frac{\dfrac{\partial D_h(\tau'';q)}{\partial t}}{1-D_h(\tau'';q)} > 0 .$$

Since $D(\tau;q)$ is increasing in $[\tau^1,1]$, any selection of time t in (τ',τ'') must be a best response for any player $j \in N$;

$$\frac{\dfrac{\partial D_j(t;q)}{\partial t}}{1-D_j(t;q)} = \min_{h\neq i} \frac{\dfrac{\partial D_h(t;q)}{\partial t}}{1-D_h(t;q)},$$

which implies $\dfrac{\partial q_j(t)}{\partial t} > 0$. Hence, the first-order condition holds; for every $t \in (\tau',\tau'')$,

$$\frac{(1-\varepsilon)q_j{}'(t)}{1-(1-\varepsilon)q_j(t)} = \frac{\overline{v}'(t)}{(n-1)\{\overline{v}(t)-\underline{v}(t)\}} .$$

However, from (7),

$$\frac{(1-\varepsilon)q_i{}'(t)}{1-(1-\varepsilon)q_i(t)} > \frac{\overline{v}'(t)}{(n-1)\{\overline{v}(t)-\underline{v}(t)\}},$$

implying that the first-order condition does not hold for player i for every $t \in (\tau',\tau'')$; instead $\dfrac{\partial u_i(\tau,q_{-i},\varepsilon)}{\partial \tau} < 0$ holds in this case. This inequality implies that player i prefers τ' to any time in $(\tau',\tau''+\varepsilon)$;

$$\frac{\partial D_i(\tau;q)}{\partial t} = 0 \quad \text{for all} \quad \tau \in (\tau',\tau''+\eta),$$

where η is positive but close to zero. This is a contradiction, because the inequality in (8) implies $\dfrac{\partial D_i(\tau'';q)}{\partial t} > 0$. Hence, we have proved that any Nash equilibrium q is symmetric.

From the above observations, we have proved Theorem 1.

Q.E.D.

We specify another symmetric strategy profile $\hat{q} = (\hat{q}_i)_{i\in N}$;

$$\hat{q}_i(0) = 1 \quad \text{for all} \quad i \in N .$$

According to $\hat{q}$, any player immediately quits the game at the initial time. We define the *overall future benefit* by

$$R \equiv \frac{\overline{v}(1)-\overline{v}(0)}{\overline{v}(0)-\underline{v}(0)}.$$

The following theorem shows a necessary and sufficient condition for $\hat{q}$ to be a Nash equilibrium; the greater the overall future benefit R is, the less likely the timing game immediately ends at the initial time.

Theorem 2: *The symmetric strategy profile* $\hat{q}$ *is a Nash equilibrium if and only if*

$$R \leq \sum_{1\leq l\leq n-1} \frac{(n-1)!}{l!(n-1-l)!}\left(\frac{1-\varepsilon}{\varepsilon}\right)^l \frac{1}{l+1}.$$

Proof: For every $t \in (0,1]$,

$$u_1(t,\hat{q}_{-1}) = \varepsilon^{n-1}\overline{v}(t) + (1-\varepsilon^{n-1})\underline{v}(0)$$
$$\leq \varepsilon^{n-1}\overline{v}(1) + (1-\varepsilon^{n-1})\underline{v}(0) = u_1(1,\hat{q}_{-1}),$$

and

$$u_1(0,\hat{q}_{-1}) = \{\sum_{1\leq l\leq n-1} \frac{(n-1)!}{l!(n-1-l)!}(1-\varepsilon)^l \varepsilon^{n-1-l}\frac{1}{l+1}\}\overline{v}(0)$$
$$+\{1-\sum_{1\leq l\leq n-1} \frac{(n-1)!}{l!(n-1-l)!}(1-\varepsilon)^l \varepsilon^{n-1-l}\frac{1}{l+1}\}\underline{v}(0).$$

Hence, a necessary and sufficient condition for $\hat{q}$ to be a Nash equilibrium is given by

$$u_1(0,\hat{q}_{-1}) = \{\sum_{1\leq l\leq n-1} \frac{(n-1)!}{l!(n-1-l)!}(1-\varepsilon)^l \varepsilon^{n-1-l}\frac{1}{l+1}\}\overline{v}(0)$$
$$+\{1-\sum_{1\leq l\leq n-1} \frac{(n-1)!}{l!(n-1-l)!}(1-\varepsilon)^l \varepsilon^{n-1-l}\frac{1}{l+1}\}\underline{v}(0)$$
$$\geq \varepsilon^{n-1}\overline{v}_1(1) + (1-\varepsilon^{n-1})\underline{v}_1(0) \geq u_1(1,\hat{q}_{-1}).$$

This inequality is equivalent to

$$\sum_{1\leq l\leq n-1} \frac{(n-1)!}{l!(n-1-l)!}\left(\frac{1-\varepsilon}{\varepsilon}\right)^l \frac{1}{l+1} \geq \frac{\overline{v}_1(1)-\overline{v}_1(0)}{\overline{v}_1(0)-\underline{v}_1(0)} = R.$$

Q.E.D.

3. Leverage-Driven Bubbles

We analyze a strategic aspect of bubbles and crashes, which we will define as a timing game with irrational types as follows. There is a company whose fundamental value is zero; it has no profitable business opportunity. Let $S(t) > 0$ denote the total share that the company has issued up to time $t \in [0,1]$. We assume that $S(t)$ is non-decreasing. The company raises funds by issuing shares. There exist n arbitrageurs. Let $S_i(t) > 0$ denote the share that arbitrageur $i \in N$ possesses at time t.

Even if the company is unproductive, its share price grows up as a phenomenon of bubble according to a exogenously given, continuous, and increasing price function $P:[0,1] \to (0,\infty)$. The bubble persists as long as the arbitrageurs continue to hold $n\phi \times 100\,\%$ or more shares in totality, where $\phi \in (0, 1/n)$. Once the arbitrageurs' total shareholding falls to less than $n\phi \times 100\,\%$, the bubble crashes immediately; its share price declines to zero, i.e., the correct fundamental value of this company. Even if no arbitrageur sells out through time, the bubble automatically crashes at the termination time 1.

It is implicit to assume that there are many noise traders with a plenty of money. At any time $t \in [0,1]$, the noise traders misperceive the fundamental value and unconsciously reinforce their misperception according to the above-mentioned price function $P:[0,1] \to (0,\infty)$. However, once the arbitrageurs' total shareholding falls to less than $n\phi \times 100\,\%$, the resultant selling pressure makes the market share price significantly lower than their misperception, which makes the noise traders aware of their misperception and the crash risk, immediately bursting the bubble.

The company is willing to raise funds as many as possible for its owner/manager's private usage, but within the limit that the resultant selling pressure bursts the bubble; if the company issues too many shares for each arbitrageur to keep his shareholding at not less than $\phi \times 100\,\%$, the resultant selling pressure bursts the bubble.

An effective method for the company to raise huge funds without causing the crash would be to encourage each arbitrageur to borrow money from the noise traders. During the bubble, the noise traders are unaware of their misperception and the crash risk, while the arbitrageurs are well aware of them. Because of this heterogeneity in awareness, each arbitrageur can enter into short-term debt contracts with these noise traders with no premium; the noise traders do not make any margin requirement to the arbitrageurs, because they are unaware of the crash risk.

Let $L \geq 1$ denote the exogenous cap of *leverage ratio*. Since any arbitrageur prefers to let his leverage ratio equal to this upper limit, he will have a debt obligation $\frac{L-1}{L}P(t)S_i(t)$ to his debt holders (noise traders). Hence, we define the personal capital $W_i(t)$ of each arbitrageur i as the market value of his shareholding minus his debt obligation:

(9) $$W_i(t) = P(t)S_i(t) - \frac{L-1}{L}P(t)S_i(t) = \frac{P(t)S_i(t)}{L}.$$

The personal capital $W_i(t)$ of arbitrageur i can be expressed by $(S_i(0), P, L)$, i.e., the combination of his shareholding at the initial time, the price function, and the leverage ratio, in the following manner. Since arbitrageur i earns a capital gain $\{P(t+\Delta) - P(t)\}S_i(t)$ from t to $t+\Delta$, his personal capital increases by this amount:

$$W_i(t+\Delta) = W_i(t) + \{P(t+\Delta) - P(t)\}S_i(t).$$

Hence,

(10) $$W_i'(t) = \lim_{\Delta \to 0}\frac{W_i(t+\Delta) - W_i(t)}{\Delta} = \lim_{\Delta \to 0}\frac{P(t+\Delta) - P(t)}{\Delta}S_i(t) = P'(t)S_i(t).$$

Moreover, from (9),

(11) $$W_i'(t) = \frac{P(t)S_i'(t) + P'(t)S_i(t)}{L}.$$

From (10) and (11), we have

$$P'(t)S_i(t) = \frac{P(t)S_i'(t) + P'(t)S_i(t)}{L},$$

that is,

$$S_i(t) = S_i(0)(\frac{P(t)}{P(0)})^{L-1}.$$

We assume that the arbitrageurs' shares are the same at the initial time;

$$S_i(0) = S_1(0) \quad \text{for all} \quad i \in \{1,...,n\}.$$

From this assumption, the arbitrageurs' shares are the same through time;

$$S_i(t) = S_1(t) \quad \text{for all} \quad i \in \{1,...,n\} \quad \text{and} \quad t \in [0,1].$$

The company had better keep the share that each arbitrageur possesses as $\phi \times 100\%$;

$$S_i(t) = S_1(t) = \phi S(t).$$

Hence, we have

$$S(t) = S(0)(\frac{P(t)}{P(0)})^{L-1}. \tag{12}$$

From (9) and (12), we can express $W_i(t)$ by $(S_i(0), P, L)$;

$$W_i(t) = \frac{\phi}{L} P(0) S(0) (\frac{P(t)}{P(0)})^L. \tag{12}$$

Based on these observations, we can model a strategic aspect of bubbles and crashes as a specification of timing game with irrational types played by n arbitrageurs. Each arbitrageur (player) $i \in N$ can win the game by selling out earlier than any other arbitrageur; he earns his personal capital $W_i(t)$. If arbitrageur i loses the timing game, he receives nothing. In this case, we assume that he is exempted from his debt obligation.

Hence, we specify the winner's payoff and loser's payoff as

$$\overline{v}(t) = W_i(t) = \frac{\phi}{L} P(0) S(0) (\frac{P(t)}{P(0)})^L,$$

and

$$\underline{v}(t) = 0.$$

We can derive the relative future benefit $R(t) = R^*(t)$ and the overall relative future benefit $R = R^*$ as

$$R^*(t) \equiv \frac{W_1'(t)}{W_1(t)} = L \frac{P'(t)}{P(t)},$$

and

$$R^* \equiv \frac{W_1(1)}{W_1(0)} - 1 = (\frac{P(1)}{P(0)})^L - 1.$$

Let $\tilde{\tau} = \tilde{\tau}^*$ denote the critical time, which is specified by

$$\varepsilon = \left(\frac{P(\tilde{\tau}^*)}{P(1)}\right)^{\frac{L}{n-1}}.$$

Note that the greater $\frac{P'(t)}{P(t)}$, i.e., the degree of the noise traders' reinforcement, the greater the relative future benefits $R^*(t)$, the overall relative future benefit R^*, and the critical time $\tilde{\tau}^*$. Hence, the stronger the noise traders' reinforcement, the greater the likelihood of the bubble to emerge and persist.

Since the model of this section depends on the leverage ratio cap L, we will write

$$R^*(t) = R^*(t,L), \quad R^* = R^*(L), \text{ and } \tilde{\tau}^* = \tilde{\tau}^*(L).$$

Clearly, we have

$$\frac{\partial R^*(t,L)}{\partial L} > 0, \quad \frac{\partial R^*(L)}{\partial L} > 0, \text{ and } \frac{\partial \tilde{\tau}^*(L)}{\partial L} > 0.$$

The greater the leverage ratio L, the greater the likelihood of the bubble to emerge and persist. Note also that

$$\lim_{L\to\infty} R^*(t,L) = \lim_{L\to\infty} R^*(L) = \lim_{L\to\infty} \theta^*(t,L) = \infty \quad \text{and} \quad \lim_{L\to\infty} \tilde{\tau}^*(L) = 1.$$

Even if the noise traders' reinforcement is insufficient, the bubble is likely to emerge and persist, provided the leverage ratio L is sufficient.

4. Crash-Contingent Claims

Throughout this study, we assume that the noise traders possess a plenty of money. To be precise, the noise-traders' total personal capital, denoted by $B(t)$, is greater than the sum of the market value of their shareholdings and their loan to the arbitrageurs;

$$B(t) > (1-n\phi)P(t)S(t) + \frac{L-1}{L} n\phi P(t)S(t) = (1-\frac{n\phi}{L})P(0)S(0)(\frac{P(t)}{P(0)})^L .$$

We denote by

$$Z(t) \equiv B(t) - (1-\frac{n\phi}{L})P(0)S(0)(\frac{P(t)}{P(0)})^L$$

their residual amount. We assume that

$$B'(t) > 0 \text{ and } Z(t) > 0 \text{ for all } t \in [0,1].$$

This section permits each arbitrageur $i \in N$ to purchase *crash-contingent claims* from noise traders, which is defined as the contractual agreement such that he (or she) can receive a promised monetary amount $Z_i(t)$ from the noise traders if and only if the bubble crashes at time t. Since the noise traders are unaware of the crash risk, any arbitrageur can purchase such crash-contingent claims for nothing.

We assume symmetry in that $Z_i(t) = Z_1(t)$ for all $i \in N$, and also assume that

$$Z_i(t) = \frac{Z(t)}{n} \text{ for all } i \in N .$$

Note that any arbitrageur never demands crash-contingent claim exceeding $\frac{Z(t)}{n}$; otherwise, the resulting demand pressure makes the price of the crash-contingent claim positive, thus making the noise traders aware of the crash risk, which bursts the bubble.

If an arbitrageur i wins the game, he can earn not only $W_i(t)$ but also $Z_i(t)$. If he loses the game, he can earns $Z_i(t)$ but is not exempted from his debt obligation. Since his debt obligation is given by $(L-1)W_i(t)$, any loser has to pay his debt holders the amount $\min[\frac{Z(t)}{n},(L-1)W_i(t)]$. For simplicity, this section assumes

$$Z(t) \geq n(L-1)W_i(t), \text{ i.e., } \min[\frac{Z(t)}{n},(L-1)W_i(t)] = (L-1)W_i(t) .$$

Since each arbitrageur's personal capital is given by (13), we specify the winner's payoff and the loser's payoff by

$$\overline{v}(t) = W_i(t) + Z_i(t)$$
$$= \frac{1}{n}\{B(t) - (1 - \frac{2n\phi}{L})P(0)S(0)(\frac{P(t)}{P(0)})^L\},$$

and

$$\underline{v}(t) = Z_i(t) - (L-1)W_i(t)$$
$$= \frac{1}{n}[B(t) - \{1 + \frac{(L-2)n\phi}{L}\}P(0)S(0)(\frac{P(t)}{P(0)})^L].$$

Hence, we can derive the relative future benefit $R(t) = R^{**}(t)$ and the overall relative future benefit $R = R^{**}$ as

$$R^{**}(t) \equiv \frac{W_1'(t) + Z_1'(t)}{LW_1(t)}$$
$$= \frac{1}{L\phi n}\{\frac{B'(t)}{P(0)S(0)(\frac{P(t)}{P(0)})^L} - (L - 2n\phi)\frac{P'(t)}{P(t)}\},$$

and

$$R^{**} \equiv \frac{W_1(1) + Z_1(1) - \{W_1(0) + Z_1(0)\}}{LW_1(0)}$$
$$= \frac{B(1) - B(0) - (1 - \frac{2n\phi}{L})P(0)S(0)\{(\frac{P(1)}{P(0)})^L - 1\}}{n\phi P(0)S(0)}.$$

Since the model of this section depends on the leverage ratio L, we will write

$$R^{**}(t) = R^{**}(t, L) \text{ and } R^{**} = R^{**}(L).$$

Clearly, both $R^{**}(t, L)$ and $R^{**}(L)$ are decreasing in L; the higher the leverage ratio L is, the less likely the bubble emerges and persists. These are in contrast to the Section 3's model; a high leverage ratio fosters the bubble when crash-contingent claims are not available, while it rather deters the bubble when crash-contingent claims are available.

The increase in L enhances future loans to arbitrageurs, which crowd out the future reserve for the crash-contingent claims, thus decreasing the relative future benefit. This drives the high leverage ratio to deter the bubble.

Clearly, we have

$$[R^*(t) > R^{**}(t)] \Leftrightarrow [n(L-1)W_1'(t) > Z'(t)],$$

and

$$[R^* > R^{**}] \Leftrightarrow [n(L-1)\{W_1(1) - W_1(0)\} > Z(1) - Z(0)].$$

If the leverage ratio L is sufficient and the residual amount $Z(t)$ does not grow sufficient, the bubble is less likely to emerge and persist with crash-contingent claim than without it. It is crucial to assume that any arbitrageur i can receive $Z_i(t)$ even if he wins the timing game. It is also crucial to assume that he is not exempted from his debt obligation even if he loses the game. These assumptions imply that the purchase of crash-contingent claims increases the difference between the winner's payoff and the loser's payoff, thus urging the purchasers to sell out his shareholding at early times.

5. Conclusion

We investigated the timing game with irrational types, which is a generalization of Matsushima (2013). We showed a (almost) necessary and sufficient condition for the uniqueness of Nash equilibrium. According to this unique Nash equilibrium, every player never quits the game at the initial time.

By applying the theoretical framework of this game, we analyzed leverage-driven bubbles in the stock market for the unproductive company; this company's stock price grows up according to an exogenously given reinforcement pattern. During the bubble, the company is willing to raise funds by issuing new shares as many as possible.

We regarded players as strategic arbitrageurs who decide whether to ride the bubble by continuing to purchase shares, or to burst the bubble by selling out their respective shareholdings. To ride the bubble, each arbitrageur borrow money from the noise traders by making use of the heterogeneity in awareness between the arbitrageurs and the noise traders concerning the crash risk and the noise traders' misperception.

We demonstrated two models, which are distinguished by whether crash-contingent claim, a version of naked credit default swap concerning the crash risk, is available. We then showed that the availability of crash-contingent claim gives a significant impact on the emergence and persistence of leverage-driven bubbles and the policy implication of leverage-ratio regulation. The availability of crash-contingent claim deters the bubble, while without crash-contingent claim, the bubble emerges and persists even if the degree of reinforcement is insufficient. Without crash-contingent claim, high leverage ratio fosters the bubble, while with crash-contingent claim, it rather deters the bubble. Hence, it would be an effective policy method that we will make crash-contingent claims available, and then make the leverage ratio cap as weak as possible.

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