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and Variances**

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Small Area Predictors with Dual Shrinkage of Means and Variances

Hiromasa Tamae* and Tatsuya Kubokawa†

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Abstract

The paper concerns small-area estimation in the Fay-Herriot type area-level model with random dispersions, which models the case that the sampling errors change from area to area. The resulting Bayes estimator shrinks both means and variances, but needs numerical computation to provide the estimates. In this paper, an approximated empirical Bayes (AEB) estimator with a closed form is suggested. The model parameters are estimated via the moment method, and the mean squared error of the AEB is estimated via the single parametric bootstrap method. The benchmarked estimator and a second-order unbiased estimator of the mean squared error are also derived.

Key words and phrases: Asymptotic approximation, benchmark, constrained Bayes, empirical Bayes, Fay-Herriot model, mean squared error, parametric bootstrap, random dispersion, second-order approximation, second-order unbiased estimate, small area estimation, variance modeling.

1 Introduction

Small area estimation (SAE) using linear mixed models has been extensively studied in the literature from both theoretical and applied points of view. For a good review and account on this topic, see Ghosh and Rao (1994), Pfeiffermann (2002), Rao (2003) and Datta (2009). Of these, the Fay-Herriot model introduced by Fay and Herriot (1979) has been used as an area-level model in SAE.

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Suppose that there are m small areas and that y_1, \dots, y_m are direct estimates of small area means. The Fay-Herriot model is described as

$$\begin{aligned} y_i | \xi_i &\sim \mathcal{N}(\xi_i, \sigma_i^2), \\ \xi_i &\sim \mathcal{N}(\mathbf{z}_i^T \boldsymbol{\beta}, \tau^2), \end{aligned}$$

where \mathbf{z}_i is a vector of auxiliary variables and $\boldsymbol{\beta}$ is an unknown vector of regression coefficients. Although σ_i^2 's are treated as known variances in the Fay-Herriot model, in practice, σ_i^2 are estimated quantities, and the resulting empirical Bayes (EB) estimators involve substantial estimation errors. To take this point into account, we suppose that statistics V_1, \dots, V_m are available for estimating σ_i^2 and that V_i/σ_i^2 has a chi-square distribution with n_i degrees of freedom. Then, Wang and Fuller (2003) provided estimators of the mean squared error (MSE) of the empirical Bayes estimators. For such variance modeling approaches, see Arora and Lahiri (1997), You and Chapman (2006), Dass, Maiti, Ren and Sinha (2012), Jiang and Nguyen (2012). Also see Maiti, Ren and Sinha (2014) and the references therein.

In the Fay-Herriot models with heteroscedastic unknown variances, each variance σ_i^2 cannot be estimated consistently based on V_i when n_i 's are bounded. This leads to the inconsistency properties of estimation procedures, namely, the empirical Bayes estimator does not converge to the Bayes estimator, and the MSE of the empirical Bayes estimator cannot be estimated consistently. To fix this difficulty, Maiti, *et al.* (2014) suggested that σ_i^2 has an inverse gamma distribution. It is interesting to point out that the resulting empirical Bayes (EB) estimator of ξ_i shrinks both means and variances. Since the EB includes integration with respect to σ_i^2 , however, the EB cannot be expressed in closed forms. Thus one needs numerical integration to provide values of the EB. Maiti, *et al.* (2014) derived a second-order unbiased estimator of the conditional mean squared error (cMSE) of the EB given (y_i, V_i) . However, one needs heavy numerical computation to provide values of the estimator of cMSE. For unconditional MSE of the EB, no computational algorithm was provided in Maiti, *et al.* (2014), because the computation may be much harder.

In this paper, we consider to approximate the Bayes estimator in the Fay-Herriot random dispersion model given in Maiti, *et al.* (2014). Approximating the joint probability density function, we suggest the approximated Bayes estimator

$$\xi_i^{AB} = \mathbf{z}_i^T \boldsymbol{\beta} + \left(1 - \frac{1}{1 + \tau^2(n_i + 1 + \alpha)/(V_i + \gamma)}\right)(y_i - \mathbf{z}_i^T \boldsymbol{\beta}),$$

where α and γ are model parameters in the distribution of σ_i^2 . Since

$$\frac{V_i + \gamma}{n_i + 1 + \alpha} = \frac{n_i + 1}{n_i + 1 + \alpha} \frac{V_i}{n_i + 1} + \left(1 - \frac{n_i + 1}{n_i + 1 + \alpha}\right) \frac{\gamma}{\alpha},$$

the estimator ξ_i^{AB} is a dual shrinkage estimator with shrinking y_i towards $\mathbf{z}_i^T \boldsymbol{\beta}$ and shrinking $V_i/(n_i + 1)$ towards γ/α . This approximation is valid in the case of large n_i , but we

want to use this closed-form estimator even for small n_i . For the purpose, we need to evaluate the estimation error. Since ξ_i^{AB} includes the model parameters β , τ^2 , α and γ , we estimate β with the generalized least squares estimator based on the approximated pdf and the other parameters τ^2 , α and γ via the moment methods. We show the consistency of the suggested estimators for the model parameters. Plugging-in the consistent estimators in ξ_i^{AB} yields the approximated empirical Bayes (AEB) estimator $\hat{\xi}_i^{AEB}$. The uncertainty of the AEB is measured via the unconditional mean squared errors (MSE), and we obtain a second-order unbiased estimator of the MSE via the single parametric bootstrap method.

In this paper, we also treat the benchmark problem. A potential difficulty of the AEB estimators $\hat{\xi}_i^{AEB}$ for small areas is that the overall estimate for a larger geographical area, which is constructed by a (weighted) sum of $\hat{\xi}_i^{AEB}$, is not necessarily equal to the corresponding direct estimate like the overall sample mean. For instance, we consider the weighted mean $\bar{y}_w = \sum_{j=1}^m w_j y_j$ for nonnegative constants w_j 's satisfying $\sum_{j=1}^m w_j = 1$. Then, we want to find predictors δ_i 's which satisfy the benchmark constraint $\sum_{j=1}^m w_j \delta_j = \bar{y}_w$. A solution of the benchmark problem is the constrained Bayes estimation suggested by Ghosh (1992) and Datta, Ghosh, Steorts and Maples (2011). Using this approach, we suggest the benchmarked predictor based on $\hat{\xi}_i^{AEB}$ given by

$$\delta_i^{CAB} = \hat{\xi}_i^{AEB} + \frac{w_i}{\sum_{j=1}^m w_j^2} \left\{ \bar{y}_w - \sum_{j=1}^m w_j \hat{\xi}_j^{AEB} \right\}.$$

A second-order unbiased estimator of the MSE of this constrained approximate Bayes estimator is derived.

The paper is organized as follows: A setup of the Fay-Herriot random dispersion model and the approximated Bayes estimator are given in Section 2. The estimators of the model parameters are also given there. In Section 3, the approximated empirical Bayes (AEB) estimator is evaluated in terms of the MSE, and the second-order unbiased estimator is suggested. The benchmark problem is discussed in Section 4. In Section 5, we investigate the performance of the proposed procedures through simulation and empirical studies. Concluding remarks are given in Section 6 and the technical proofs are given in the Appendix.

2 Area-level Model and Estimation of Model Parameters

2.1 Fay-Herriot random dispersion model and an approximated predictor

For m small areas, let $(y_i, V_i/n_i)$ be the pair of mean estimate and variance estimate for the i -th small area, $i = 1, \dots, m$, where n_i is degrees of freedom. Suppose that there exist

$p - 1$ covariates which are denoted by $\mathbf{z}_i = (z_{i1}, \dots, z_{ip})$ with $z_{i1} = 1$. Then we consider the following heteroscedastic area-level model with random dispersions:

$$\begin{aligned} y_i \mid \xi_i, \sigma_i^2 &\sim \mathcal{N}(\xi_i, \sigma_i^2), \\ \xi_i &\sim \mathcal{N}(\mathbf{z}_i^T \boldsymbol{\beta}, \tau^2), \\ V_i / \sigma_i^2 \mid \sigma_i^2 &\sim \chi_{n_i}^2, \\ \sigma_i^{-2} &\sim Ga(\alpha/2, 2/\gamma), \end{aligned} \quad (2.1)$$

where $(y_1, V_1), \dots, (y_m, V_m)$ are mutually independent. We call it the Fay-Herriot Random Dispersion model (hereafter, FHRD model). Here, $Ga(\alpha/2, 2/\gamma)$ denotes a gamma distribution with mean α/γ and variance $2\alpha/\gamma^2$. The unknown parameters are denoted by $\boldsymbol{\omega} = (\boldsymbol{\beta}^T, \tau^2, \alpha, \gamma)^T$ for $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^T$.

Let $\eta_i = 1/\sigma_i^2$ and $C_i = 1/[2\pi 2^{n_i/2} \Gamma(n_i/2)]$. The joint pdf of $(y_i, V_i, \xi_i, \eta_i)$ is

$$\begin{aligned} f_i(y_i, V_i, \xi_i, \eta_i) &= C_i \frac{(\gamma/2)^{\alpha/2}}{\tau \Gamma(\alpha/2)} V_i^{n_i/2-1} \eta_i^{(n_i+1+\alpha)/2-1} \\ &\times \exp \left[-\frac{\eta_i}{2} \{(y_i - \xi_i)^2 + V_i + \gamma\} - \frac{1}{2\tau^2} (\xi_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 \right]. \end{aligned} \quad (2.2)$$

It is noted that

$$\begin{aligned} &\eta_i (y_i - \xi_i)^2 + \tau^{-2} (\xi_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 + \eta_i (V_i + \gamma) \\ &= (\eta_i + \tau^{-2}) \{ \xi_i - \xi_i^M(\eta_i) \}^2 + \eta_i \left\{ V_i + \gamma + \frac{1}{\tau^2 \eta_i + 1} (y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 \right\}, \end{aligned}$$

where

$$\xi_i^M(\eta_i) = \mathbf{z}_i^T \boldsymbol{\beta} + \left(1 - \frac{1}{\tau^2 \eta_i + 1} \right) (y_i - \mathbf{z}_i^T \boldsymbol{\beta}). \quad (2.3)$$

Then, the Bayes estimator of ξ_i is described as

$$\xi_i^B = E[\xi_i \mid y_i, V_i] = \mathbf{z}_i^T \boldsymbol{\beta} + \left(1 - E \left[\frac{1}{\tau^2 \eta_i + 1} \mid y_i, V_i \right] \right) (y_i - \mathbf{z}_i^T \boldsymbol{\beta}), \quad (2.4)$$

where

$$E \left[\frac{1}{\tau^2 \eta_i + 1} \mid y_i, V_i \right] = \frac{\int_0^\infty (\tau^2 \eta_i + 1)^{-1} f_i(y_i, V_i, \eta_i) d\eta_i}{\int_0^\infty f_i(y_i, V_i, \eta_i) d\eta_i}, \quad (2.5)$$

for the marginal pdf of $f_i(y_i, V_i, \eta_i)$ given by

$$\begin{aligned} f_i(y_i, V_i, \eta_i) &= C_i \frac{(\gamma/2)^{\alpha/2}}{\Gamma(\alpha/2)} V_i^{n_i/2-1} \eta_i^{(n_i+1+\alpha)/2-1} \frac{\sqrt{2\pi}}{\sqrt{\tau^2 \eta_i + 1}} \\ &\times \exp \left[-\frac{\eta_i}{2} \left\{ V_i + \gamma + \frac{1}{\tau^2 \eta_i + 1} (y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 \right\} \right]. \end{aligned} \quad (2.6)$$

When $\sigma_i^2 = 1/\eta_i$ is fixed and unknown, it may be estimated with V_i/n_i . Then from (2.3), one gets the estimator

$$\mathbf{z}_i^T \boldsymbol{\beta} + \left(1 - \frac{1}{\tau^2(n_i/V_i) + 1}\right)(y_i - \mathbf{z}_i^T \boldsymbol{\beta}),$$

which is not very stable for small n_i due to the estimation error in V_i/n_i . The Bayes estimator (2.4) can fix this undesirable property. However, we resort to numerical integration to obtain the Bayes estimator and the empirical Bayes estimator. It may be computationally harder to evaluate the mean squared error of the empirical Bayes estimator.

We want to suggest another estimator with a closed form. To this end, we begin by integrating out the joint density (2.2) with respect to η_i . Then the marginal pdf of (y_i, V_i, ξ_i) is written as

$$\begin{aligned} h_i(y_i, V_i, \xi_i) &= C_i \frac{(\gamma/2)^{\alpha/2}}{\tau \Gamma(\alpha/2)} V_i^{n_i/2-1} \exp \left[-\frac{1}{2\tau^2} (\xi_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 \right] \\ &\quad \times \Gamma \left(\frac{n_i + 1 + \alpha}{2} \right) \left(\frac{2}{(y_i - \xi_i)^2 + V_i + \gamma} \right)^{(n_i+1+\alpha)/2}. \end{aligned}$$

Based on the density $h_i(y_i, V_i, \xi_i)$, the Bayes estimator of ξ_i is also expressed as $\xi_i^B = E[\xi_i | y_i, V_i]$. We here consider to approximate the marginal pdf $h_i(y_i, V_i, \xi_i)$. It is noted that

$$\begin{aligned} &\left(\frac{2}{(y_i - \xi_i)^2 + V_i + \gamma} \right)^{(n_i+1+\alpha)/2} \\ &= \left(\frac{2}{V_i + \gamma} \right)^{(n_i+1+\alpha)/2} \exp \left[-\frac{n_i + 1 + \alpha}{2} \log \left(1 + \frac{(y_i - \xi_i)^2}{V_i + \gamma} \right) \right], \end{aligned}$$

Then, the function $\log\{1 + (y_i - \xi_i)^2/(V_i + \gamma)\}$ is approximated as

$$\log\{1 + (y_i - \xi_i)^2/(V_i + \gamma)\} \approx (y_i - \xi_i)^2/(V_i + \gamma). \quad (2.7)$$

This approximation can be guaranteed when n_i is large. However, we use this approximation for small n_i as well, and derive estimators of the unknown parameters and predictors for ξ_i based on this approximation.

Using this approximation, we can rewrite the pdf $h_i(y_i, V_i, \xi_i)$ as

$$\begin{aligned} h_i^*(y_i, V_i, \xi_i) &= C_i \frac{(\gamma/2)^{\alpha/2}}{\tau \Gamma(\alpha/2)} V_i^{n_i/2-1} \left(\frac{2}{V_i + \gamma} \right)^{(n_i+1+\alpha)/2} \Gamma \left(\frac{n_i + 1 + \alpha}{2} \right) \\ &\quad \times \exp \left[-\frac{1}{2\tau^2} (\xi_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 - \frac{A_i}{2} (y_i - \xi_i)^2 \right], \end{aligned}$$

for $A_i = (n_i + 1 + \alpha)/(V_i + \gamma)$. It is noted that $h_i^*(y_i, V_i, \xi_i)$ is not a pdf. Since

$$\frac{1}{\tau^2} (\xi_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 + A_i (y_i - \xi_i)^2 = \frac{1 + \tau^2 A_i}{\tau^2} \left(\xi_i - \frac{\mathbf{z}_i^T \boldsymbol{\beta} + \tau^2 A_i y_i}{1 + \tau^2 A_i} \right)^2 + \frac{A_i}{1 + \tau^2 A_i} (y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2,$$

we get the approximated Bayes estimator of ξ_i given by

$$\xi_i^{AB} = \xi_i^{AB}(\boldsymbol{\beta}, \boldsymbol{\theta}) = \frac{\mathbf{z}_i^T \boldsymbol{\beta} + \tau^2 A_i y_i}{1 + \tau^2 A_i} = \mathbf{z}_i^T \boldsymbol{\beta} + (1 - B_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta}), \quad (2.8)$$

where $\boldsymbol{\theta} = (\tau^2, \alpha, \gamma)^T$ and

$$B_i = B_i(\boldsymbol{\theta}, V_i) = \frac{1}{1 + \tau^2 A_i} = \frac{1}{1 + \tau^2(n_i + 1 + \alpha)/(V_i + \gamma)}.$$

It is noted that this is not the Bayes estimator, but the approximated Bayes estimator when the approximation (2.7) is valid. Since the approximated Bayes estimator has a simple and reasonable form, however, we shall use this estimator even if this approximation is not appropriate. The following proposition implies that the approximated Bayes estimator ξ_i^{AB} has less shrinkage than the Bayes estimator ξ_i^B given in (2.4). The proof is given in the Appendix.

Proposition 2.1 *The shrinkage function B_i in ξ_i^{AB} is less than the shrinkage function (2.5) in the Bayes estimator ξ_i^B , namely,*

$$E\left[\frac{1}{\tau^2 \eta_i + 1} \mid y_i, V_i\right] \geq B_i = \frac{1}{1 + \tau^2(n_i + 1 + \alpha)/(V_i + \gamma)}.$$

2.2 Estimation of the model parameters

We now provide estimators of the model parameters $\boldsymbol{\beta}$, τ^2 , α and γ .

[1] **Estimation of $\boldsymbol{\beta}$.** Integrating out $h_i^*(y_i, V_i, \xi_i)$ with respect to ξ_i , we have

$$\begin{aligned} h_i^*(y_i, V_i) &= C_i \frac{(\gamma/2)^{\alpha/2}}{\Gamma(\alpha/2)} V_i^{n_i/2-1} \left(\frac{2}{V_i + \gamma}\right)^{(n_i+1+\alpha)/2} \Gamma\left(\frac{n_i + 1 + \alpha}{2}\right) \\ &\quad \times \sqrt{2\pi} \sqrt{B_i} \exp\left[-\frac{1}{2\tau^2}(1 - B_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2\right]. \end{aligned}$$

Let $\ell^* = \ell^*(\boldsymbol{\beta}, \tau^2, \alpha, \gamma) = \sum_{i=1}^m \ell_i^*$ for $\ell_i^* = \log h_i^*(y_i, V_i)$. Since

$$2 \frac{\partial \ell^*}{\partial \boldsymbol{\beta}} = \frac{1}{\tau^2} \sum_{i=1}^m (1 - B_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta}) \mathbf{z}_i,$$

we get the estimator

$$\tilde{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}(\boldsymbol{\beta}, \tau^2, \alpha, \gamma) = \left(\sum_{j=1}^m (1 - B_j) \mathbf{z}_j \mathbf{z}_j^T\right)^{-1} \sum_{j=1}^m (1 - B_j) \mathbf{z}_j y_j, \quad (2.9)$$

which is the generalized least squares (GLS) estimator of $\boldsymbol{\beta}$.

[2] Estimation of τ^2 . To estimate τ^2 , we consider the expectation $E[(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 / (V_i + \gamma)]$. The conditional expectation of $(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2$ given V_i is decomposed as

$$E[(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 | V_i] = E[(y_i - \xi_i)^2 | V_i] + 2E[(y_i - \xi_i)(\xi_i - \mathbf{z}_i^T \boldsymbol{\beta}) | V_i] + E[(\xi_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 | V_i].$$

Since $E[(y_i - \xi_i)^2 | V_i, \eta_i, \xi_i] = 1/\eta_i$, $E[(y_i - \xi_i)(\xi_i - \mathbf{z}_i^T \boldsymbol{\beta}) | V_i, \eta_i, \xi_i] = 0$ and $E[(\xi_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 | V_i, \eta_i] = \tau^2$, it is seen that

$$E[(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 | V_i] = E[\eta_i^{-1} | V_i] + \tau^2.$$

The joint pdf of (V_i, η_i) is

$$f_i(V_i, \eta_i) = \frac{\gamma^{\alpha/2} V_i^{n_i/2-1}}{\Gamma(n_i/2) \Gamma(\alpha/2) 2^{(n_i+\alpha)/2}} \eta_i^{(n_i+\alpha)/2-1} e^{-(\eta_i/2)(V_i+\gamma)}, \quad (2.10)$$

so that the marginal pdf of V_i is

$$f_i(V_i) = \frac{\Gamma((n_i + \alpha)/2)}{\Gamma(n_i/2) \Gamma(\alpha/2)} \frac{\gamma^{\alpha/2} V_i^{n_i/2-1}}{(V_i + \gamma)^{(n_i+\alpha)/2}}, \quad (2.11)$$

and the conditional pdf of η_i given V_i is

$$\eta_i | V_i \sim Ga\left(\frac{n_i + \alpha}{2}, \frac{2}{V_i + \gamma}\right). \quad (2.12)$$

Thus, one gets

$$E[\eta_i^{-1} | V_i] = \frac{V_i + \gamma}{n_i + \alpha - 2},$$

which implies that

$$E[(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 | V_i] = \frac{V_i + \gamma}{n_i + \alpha - 2} + \tau^2. \quad (2.13)$$

Thus, from this equality, we consider the moment $E[(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 / (V_i + \gamma)]$, which is

$$E\left[\frac{(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2}{V_i + \gamma}\right] = \frac{1}{n_i + \alpha - 2} + E\left[\frac{\tau^2}{V_i + \gamma}\right].$$

To calculate the moments of V_i from the marginal pdf (2.11), the following equality is useful: In general, for real numbers ℓ and k , it can be shown that

$$E\left[\frac{V_i^\ell}{(V_i + \gamma)^k}\right] = \frac{\Gamma((n_i + \alpha)/2)}{\Gamma((n_i + \alpha)/2 + k)} \frac{\Gamma(n_i/2 + \ell)}{\Gamma(n_i/2)} \frac{\Gamma(\alpha/2 + k - \ell)}{\Gamma(\alpha/2)} \gamma^{\ell-k}. \quad (2.14)$$

For $\ell = 0$ and $k = 1$, we have $E[1/(V_i + \gamma)] = \alpha/\{\gamma(n_i + \alpha)\}$, so that

$$E\left[\frac{(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2}{V_i + \gamma}\right] = \frac{1}{n_i + \alpha - 2} + \frac{\alpha/\gamma}{n_i + \alpha} \tau^2.$$

When α and γ are known and β is estimated by the ordinary least squares (OLS) estimator $\widehat{\beta}_{OLS} = (\sum_{j=1}^m \mathbf{z}_j \mathbf{z}_j^T)^{-1} \sum_{j=1}^m \mathbf{z}_j y_j$, this gives us the estimator

$$\hat{\tau}^2 = \left(\sum_{i=1}^m \frac{\alpha/\gamma}{n_i + \alpha} \right)^{-1} \sum_{i=1}^m \left\{ \frac{(y_i - \mathbf{z}_i^T \widehat{\beta}_{OLS})^2}{V_i + \gamma} - \frac{1}{n_i + \alpha - 2} \right\}. \quad (2.15)$$

[3] Estimation of α . Concerning the estimation of α , we concentrate on the marginal pdf (2.11) of V_i . Since $\log f_i(V_i)$ is expressed as

$$\log f_i(V_i) = \log \Gamma\left(\frac{n_i + \alpha}{2}\right) - \log \Gamma\left(\frac{n_i}{2}\right) - \log \Gamma\left(\frac{\alpha}{2}\right) + \frac{\alpha}{2} \log \gamma + \frac{n_i - 2}{2} \log V_i - \frac{n_i + \alpha}{2} \log(V_i + \gamma),$$

we have

$$2 \frac{\partial}{\partial \alpha} \log f_i(V_i) = \psi\left(\frac{n_i + \alpha}{2}\right) - \psi\left(\frac{\alpha}{2}\right) + \log \gamma - \log(V_i + \gamma),$$

where $\psi(\cdot)$ is the digamma function given by $\psi(x) = \Gamma'(x)/\Gamma(x)$. Since $E[\partial f_i(V_i)/\partial \alpha] = 0$, one gets

$$E[\log(V_i + \gamma)] = \psi\left(\frac{n_i + \alpha}{2}\right) - \psi\left(\frac{\alpha}{2}\right) + \log \gamma. \quad (2.16)$$

We here note the following equality. For real numbers ℓ and k , it can be shown that

$$\begin{aligned} E\left[\frac{V_i^\ell}{(V_i + \gamma)^k} \log(V_i + \gamma)\right] &= \frac{\Gamma((n_i + \alpha)/2)}{\Gamma((n_i + \alpha)/2 + k)} \frac{\Gamma(n_i/2 + \ell)}{\Gamma(n_i/2)} \frac{\Gamma(\alpha/2 + k - \ell)}{\Gamma(\alpha/2)} \gamma^{\ell-k} \\ &\quad \times \left\{ \psi\left(\frac{n_i + \alpha}{2} + k\right) - \psi\left(\frac{\alpha}{2} + k - \ell\right) + \log \gamma \right\}. \end{aligned} \quad (2.17)$$

For $\ell = 1$ and $k = 1$, we have

$$E\left[\frac{V_i}{V_i + \gamma} \log(V_i + \gamma)\right] = \frac{n_i}{n_i + \alpha} \left\{ \psi\left(\frac{n_i + \alpha}{2} + 1\right) - \psi\left(\frac{\alpha}{2}\right) + \log \gamma \right\}.$$

Since the digamma function has the property that $\psi(x + 1) = \psi(x) + 1/x$, it follows from (2.16) that

$$E\left[\frac{V_i}{V_i + \gamma} \log(V_i + \gamma)\right] = \frac{n_i}{n_i + \alpha} \left\{ E[\log(V_i + \gamma)] + \frac{2}{n_i + \alpha} \right\}. \quad (2.18)$$

This can be rewritten as

$$\begin{aligned} \alpha^2 E\left[\frac{V_i}{V_i + \gamma} \log(V_i + \gamma)\right] + \alpha E\left[n_i \frac{V_i - \gamma}{V_i + \gamma} \log(V_i + \gamma)\right] \\ - n_i^2 E\left[\frac{\gamma}{V_i + \gamma} \log(V_i + \gamma)\right] - 2n_i = 0, \end{aligned}$$

which yields an estimator of α . In fact, we can suggest the estimator as the solution of the quadratic equation

$$\alpha^2 \sum_{i=1}^m \frac{V_i}{V_i + \gamma} \log(V_i + \gamma) + \alpha \sum_{i=1}^m n_i \frac{V_i - \gamma}{V_i + \gamma} \log(V_i + \gamma) - \sum_{i=1}^m n_i \left\{ \frac{n_i \gamma}{V_i + \gamma} \log(V_i + \gamma) + 2 \right\} = 0. \quad (2.19)$$

[4] **Estimation of γ .** Concerning the estimation of γ , from (2.14), it follows that

$$E\left[\frac{V_i}{V_i + \gamma}\right] = \frac{n_i}{n_i + \alpha}.$$

Thus, one gets the estimator of γ as the solution of the equation

$$\sum_{i=1}^m \frac{V_i}{V_i + \gamma} = \sum_{i=1}^m \frac{n_i}{n_i + \alpha}. \quad (2.20)$$

3 Evaluation of Uncertainty of Prediction

Substituting the estimators of $\boldsymbol{\beta}$, τ^2 , α and γ into (2.8), we get the predictor

$$\widehat{\xi}_i^{AEB} = \xi_i^{AB}(\widehat{\boldsymbol{\beta}}, \widehat{\tau}^2, \widehat{\alpha}, \widehat{\gamma}) = \mathbf{z}_i^T \widehat{\boldsymbol{\beta}} + (1 - \widehat{B}_i)(y_i - \mathbf{z}_i^T \widehat{\boldsymbol{\beta}}), \quad (3.1)$$

where

$$\widehat{B}_i = B_i(\widehat{\tau}^2, \widehat{\alpha}, \widehat{\gamma}) = \frac{1}{1 + \widehat{\tau}^2(n_i + 1 + \widehat{\alpha})/(V_i + \widehat{\gamma})}. \quad (3.2)$$

$$\widehat{\boldsymbol{\beta}} = \left(\sum_{j=1}^m (1 - \widehat{B}_j) \mathbf{z}_j \mathbf{z}_j^T \right)^{-1} \sum_{j=1}^m (1 - \widehat{B}_j) \mathbf{z}_j y_j, \quad (3.3)$$

We call it the approximated empirical Bayes estimator. It is noted that the term $(V_i + \widehat{\gamma})/(n_i + 1 + \widehat{\alpha})$ in \widehat{B}_i is expressed as

$$\frac{V_i + \widehat{\gamma}}{n_i + 1 + \widehat{\alpha}} = \frac{n_i + 1}{n_i + 1 + \widehat{\alpha}} \frac{V_i}{n_i + 1} + \left(1 - \frac{n_i + 1}{n_i + 1 + \widehat{\alpha}}\right) \frac{\widehat{\gamma}}{\widehat{\alpha}},$$

which shrinks $V_i/(n_i + 1)$ towards the target $\widehat{\gamma}/\widehat{\alpha}$. Thus, the predictor $\widehat{\xi}_i^{AEB}$ is a double shrinkage procedure such that y_i and $V_i/(n_i + 1)$ are shrunk towards \mathbf{z}_i^T and $\widehat{\gamma}/\widehat{\alpha}$, respectively. In this section, we derive a second-order unbiased estimator of the mean squared error (MSE) of $\widehat{\xi}_i^{AEB}$.

We begin by rewriting the predictor as

$$\widehat{\xi}_i^{AEB} = \{(1 - B_i)y_i + B_i \mathbf{z}_i^T \boldsymbol{\beta}\} - \{(\widehat{B}_i - B_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta}) - \widehat{B}_i \mathbf{z}_i^T (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})\}.$$

Thus, the MSE of $\widehat{\xi}_i^{AEB}$ is decomposed as

$$MSE(\widehat{\xi}_i^{AEB}) = E[(\widehat{\xi}_i - \xi_i)^2] = g_1 + g_2 - 2g_3, \quad (3.4)$$

where

$$\begin{aligned} g_1 &= E[\{(1 - B_i)y_i + B_i\mathbf{z}_i^T\boldsymbol{\beta} - \xi_i\}^2], \\ g_2 &= E[\{(\widehat{B}_i - B_i)(y_i - \mathbf{z}_i^T\boldsymbol{\beta}) - \widehat{B}_i\mathbf{z}_i^T(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})\}^2], \\ g_3 &= E[\{(1 - B_i)y_i + B_i\mathbf{z}_i^T\boldsymbol{\beta} - \xi_i\}\{(\widehat{B}_i - B_i)(y_i - \mathbf{z}_i^T\boldsymbol{\beta}) - \widehat{B}_i\mathbf{z}_i^T(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})\}]. \end{aligned} \quad (3.5)$$

To evaluate g_1 , g_2 and g_3 , we use the following theorem under the assumption (A): For notational simplicity, let $\boldsymbol{\theta} = (\tau^2, \alpha, \gamma)^T$ and $\widehat{\boldsymbol{\theta}} = (\widehat{\tau}^2, \widehat{\alpha}, \widehat{\gamma})^T$. Also, let $\boldsymbol{\omega} = (\boldsymbol{\beta}^T, \boldsymbol{\theta}^T)^T$ and $\widehat{\boldsymbol{\omega}} = (\widehat{\boldsymbol{\beta}}^T, \widehat{\boldsymbol{\theta}}^T)^T$.

Assumption (A)

(A1) There exist \underline{n} and \bar{n} such that $\underline{n} \leq n_i \leq \bar{n}$ for $i = 1, \dots, m$. The dimension p is bounded.

(A2) The matrix $m^{-1} \sum_{i=1}^m \mathbf{z}_i \mathbf{z}_i^T$ converges to a positive definite matrix.

Theorem 3.1 *Assume the condition (A) and $n_i + \alpha > 4$ for $i = 1, \dots, m$. Then, $E[\widehat{\boldsymbol{\omega}} - \boldsymbol{\omega}] = O(m^{-1})$ and $E[(\widehat{\boldsymbol{\omega}} - \boldsymbol{\omega})(\widehat{\boldsymbol{\omega}} - \boldsymbol{\omega})^T] = O(m^{-1})$. Also, the conditional moments given y_i, V_i satisfy that $E[\widehat{\boldsymbol{\omega}} - \boldsymbol{\omega} \mid y_i, V_i] = E[\widehat{\boldsymbol{\omega}} - \boldsymbol{\omega}] + o_p(m^{-1})$ and $E[(\widehat{\boldsymbol{\omega}} - \boldsymbol{\omega})(\widehat{\boldsymbol{\omega}} - \boldsymbol{\omega})^T \mid y_i, V_i] = E[(\widehat{\boldsymbol{\omega}} - \boldsymbol{\omega})(\widehat{\boldsymbol{\omega}} - \boldsymbol{\omega})^T] + o_p(m^{-1})$.*

We first evaluate g_1 . Since $(1 - B_i)y_i + B_i\mathbf{z}_i^T\boldsymbol{\beta} - \xi_i = (1 - B_i)(y_i - \mathbf{z}_i^T\boldsymbol{\beta}) - (\xi_i - \mathbf{z}_i^T\boldsymbol{\beta})$ and B_i is a function of V_i , it is seen that

$$\begin{aligned} g_1 &= E[(1 - B_i)^2(y_i - \mathbf{z}_i^T\boldsymbol{\beta})^2] + E[(\xi_i - \mathbf{z}_i^T\boldsymbol{\beta})^2] - 2E[(1 - B_i)(y_i - \mathbf{z}_i^T\boldsymbol{\beta})(\xi_i - \mathbf{z}_i^T\boldsymbol{\beta})] \\ &= E[(1 - B_i)^2(\sigma_i^2 + \tau^2) + \tau^2 - 2(1 - B_i)\tau^2]. \end{aligned}$$

Note that $E[\sigma_i^2 \mid V_i] = E[1/\eta_i \mid V_i] = (V_i + \gamma)/(n_i - 2 + \alpha)$. Thus, one gets $g_1 = E[G(\boldsymbol{\theta}, V_i)]$, where

$$G(\boldsymbol{\theta}, V_i) = \frac{V_i + \gamma}{n_i - 2 + \alpha} (1 - B_i)^2 + \tau^2 B_i^2, \quad (3.6)$$

which is of order $O_p(1)$. We here rewrite g_1 as

$$g_1 = E[G(\widehat{\boldsymbol{\theta}}, V_i)] - E[G(\widehat{\boldsymbol{\theta}}, V_i) - G(\boldsymbol{\theta}, V_i)] = g_{11} - g_{12}. \quad (\text{say})$$

An exact unbiased estimator of g_{11} is $G(\widehat{\boldsymbol{\theta}}, V_i)$. Concerning g_{12} , the Taylor series expansion give us the approximation as

$$g_{12} = E\left[(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T \frac{\partial}{\partial \boldsymbol{\theta}} G(\boldsymbol{\theta}, V_i)\right] + \frac{1}{2} E\left[(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T \left(\frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} G(\boldsymbol{\theta}, V_i)\right) (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})\right] + O(m^{-3/2}).$$

Since $\widehat{\boldsymbol{\omega}} - \boldsymbol{\omega} = O_p(m^{-1/2})$ from Theorem 3.1, it is clear that the second term in g_{12} is of order $O(m^{-1})$. For the first term, it follows from Theorem 3.1 that

$$\begin{aligned} E\left[(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T \frac{\partial}{\partial \boldsymbol{\theta}} G(\boldsymbol{\theta}, V_i) \mid y_i, V_i\right] &= E[(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T \mid y_i, V_i] \frac{\partial}{\partial \boldsymbol{\theta}} G(\boldsymbol{\theta}, V_i) \\ &= E[(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T] \frac{\partial}{\partial \boldsymbol{\theta}} G(\boldsymbol{\theta}, V_i) + o_p(m^{-1}), \end{aligned}$$

which is of order $O_p(m^{-1})$. This shows that $g_{12} = O(m^{-1})$.

For g_2 , it is clear that $g_2 = O(m^{-1})$. For g_3 , it is noted that

$$\begin{aligned} g_3 &= E\left[\{(1 - B_i)y_i + B_i \mathbf{z}_i^T \boldsymbol{\beta} - E[\xi_i \mid y_i, V_i]\} \{(\widehat{B}_i - B_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta}) - B_i \mathbf{z}_i^T (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})\}\right] \\ &\quad - E\left[\{(1 - B_i)y_i + B_i \mathbf{z}_i^T \boldsymbol{\beta} - E[\xi_i \mid y_i, V_i]\} (\widehat{B}_i - B_i) \mathbf{z}_i^T (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta})\right]. \end{aligned}$$

The second term is of order $O(m^{-1})$. For the first term,

$$E\left[\{(1 - B_i)y_i + B_i \mathbf{z}_i^T \boldsymbol{\beta} - E[\xi_i \mid y_i, V_i]\} \{E[\widehat{B}_i - B_i \mid y_i, V_i](y_i - \mathbf{z}_i^T \boldsymbol{\beta}) - B_i \mathbf{z}_i^T E[\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} \mid y_i, V_i]\}\right],$$

which is approximated as

$$E\left[\{(1 - B_i)y_i + B_i \mathbf{z}_i^T \boldsymbol{\beta} - E[\xi_i \mid y_i, V_i]\} \{E[\widehat{B}_i - B_i](y_i - \mathbf{z}_i^T \boldsymbol{\beta}) - B_i \mathbf{z}_i^T E[\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}]\}\right] + o(m^{-1}).$$

This shows that $g_3 = O(m^{-1})$. Hence, we get the following proposition.

Proposition 3.1 *Assume the condition (A) and $n_i + \alpha > 4$ for $i = 1, \dots, m$. Then, the MSE of the predictor $\widehat{\xi}_i^{AEB}$ is decomposed as*

$$MSE(\widehat{\xi}_i^{AEB}) = g_{11} + \{-g_{12} + g_2 - 2g_3\},$$

where $g_{11} = O(1)$, $g_{12} = O(m^{-1})$, $g_2 = O(m^{-1})$ and $g_3 = O(m^{-1})$.

We next estimate the MSE of $\widehat{\xi}_i^{AEB}$. An exact unbiased estimator of g_{11} is given by

$$\widehat{g}_{11} = G(\widehat{\boldsymbol{\theta}}, V_i) = \frac{V_i + \widehat{\gamma}}{n_i - 2 + \widehat{\alpha}} (1 - \widehat{B}_i)^2 + \widehat{\tau}^2 \widehat{B}_i^2.$$

To provide second-order unbiased estimators of g_{12} , g_2 and g_3 , we use the parametric bootstrap method. Let (y_i^*, V_i^*) , $i = 1, \dots, m$, be a bootstrap sample generated from the model:

$$\begin{aligned} y_i^* \mid \xi_i^*, \eta_i^* &\sim \mathcal{N}(\xi_i^*, 1/\eta_i^*), \\ \xi_i^* &\sim \mathcal{N}(\mathbf{z}_i^{*T} \widehat{\boldsymbol{\beta}}, \widehat{\tau}^2), \\ V_i^* \eta_i^* \mid \eta_i^* &\sim \chi_{n_i}^2, \\ \eta_i^* &\sim Ga(\widehat{\alpha}/2, 2/\widehat{\gamma}), \end{aligned} \tag{3.7}$$

where $\widehat{\boldsymbol{\beta}}$, $\widehat{\tau}^2$, $\widehat{\alpha}$ and $\widehat{\gamma}$ are estimators constructed from the original model (2.1). The bootstrap estimators $\widehat{\boldsymbol{\beta}}^*$, $\widehat{\tau}^*$, $\widehat{\alpha}^*$ and $\widehat{\gamma}^*$ are calculated via the same manner as in $\widehat{\boldsymbol{\beta}}$, $\widehat{\tau}^2$, $\widehat{\alpha}$

and $\hat{\gamma}$ except that the bootstrap estimators are calculated based on (y_i^*, V_i^*) 's instead of (y_i, V_i) 's. Then we can estimate g_{12} , g_2 and g_3 with

$$\begin{aligned} g_{12}^* &= E^*[G(\hat{\boldsymbol{\theta}}^*, V_i^*) - G(\hat{\boldsymbol{\theta}}, V_i^*)], \\ g_2^* &= E^*[\{(\hat{B}_i^* - B_i^*)(y_i^* - \mathbf{z}_i^T \hat{\boldsymbol{\beta}}) - \hat{B}_i^* \mathbf{z}_i^T (\hat{\boldsymbol{\beta}}^* - \hat{\boldsymbol{\beta}})\}^2], \\ g_3^* &= E^*[\{(1 - B_i^*)y_i^* + B_i^* \mathbf{z}_i^T \hat{\boldsymbol{\beta}} - \xi_i^*\} \{(\hat{B}_i^* - B_i^*)(y_i^* - \mathbf{z}_i^T \hat{\boldsymbol{\beta}}) - \hat{B}_i^* \mathbf{z}_i^T (\hat{\boldsymbol{\beta}}^* - \hat{\boldsymbol{\beta}})\}], \end{aligned} \quad (3.8)$$

where $\hat{\boldsymbol{\theta}}^* = (\hat{\tau}^*, \hat{\alpha}^*, \hat{\gamma}^*)^T$, $B_i^* = B_i(\hat{\boldsymbol{\theta}}, V_i^*)$ and $\hat{B}_i^* = B_i(\hat{\boldsymbol{\theta}}^*, V_i^*)$. It can be seen that these are second-order unbiased estimators.

Proposition 3.2 *Assume the condition (A) and $n_i + \alpha > 4$ for $i = 1, \dots, m$. Then, a second-order unbiased estimator of the MSE of $\hat{\xi}_i^{AEB}$ is*

$$mse(\hat{\xi}_i^{AEB}) = \hat{g}_{11} + \{-g_{12}^* + g_2^* - 2g_3^*\}, \quad (3.9)$$

where $E[\hat{g}_{11}] = g_{11}$, $E[g_{12}^*] = g_{12} + o(m^{-1})$, $E[g_2^*] = g_2 + o(m^{-1})$ and $E[g_3^*] = g_3 + o(m^{-1})$.

4 Benchmarked Prediction

In this section, we consider the benchmark problem which imposes a constraint on predictors for small areas. The benchmarked predictors are derived and an approximated unbiased estimator of their MSE is provided.

Although the predictors $\hat{\xi}_i$ in (3.1) give reliable estimates for ξ_i by borrowing strength from the surrounding areas, we are faced with a potential difficulty of the predictor. That is, the overall estimate for a larger geographical area, which is constructed by a (weighted) sum of $\hat{\xi}_i$, is not necessarily equal to the corresponding direct estimate like the overall sample mean. To describe it specifically, let w_j 's be nonnegative constants such that $\sum_{j=1}^m w_j = 1$. Suppose that the mean of the total areas is estimated by the weighted sum of y_j 's, $\sum_{j=1}^m w_j y_j$. Then, the benchmark problem is described as an issue of finding estimators δ_j such that

$$\sum_{j=1}^m w_j \delta_j = \sum_{j=1}^m w_j y_j \equiv \bar{y}_w. \quad (4.1)$$

A solution of the benchmark problem is the constrained Bayes estimation suggested by Ghosh (1992) and Datta, Ghosh, Steorts and Maples (2011), who considered the minimization of $\sum_{i=1}^m E[(\delta_i - \xi_i)^2 \mid Data]$ under the constraint (4.1). Using the Lagrange multiplier method, one gets the constrained Bayes estimator

$$\delta_i^{CB} = E[\xi_i \mid y_i, V_i] + \frac{w_i}{\sum_{j=1}^m w_j^2} \left\{ \bar{y}_w - \sum_{j=1}^m w_j E[\xi_j \mid y_j, V_j] \right\}.$$

Since the Bayes estimator $E[\xi_i | y_i, V_i]$ cannot be expressed in a closed form, we replace it with the approximated empirical Bayes estimator $\hat{\xi}_i$ given in (3.1). The resulting benchmarked predictor is

$$\delta_i^{CAB} = \hat{\xi}_i^{AEB} + \frac{w_i}{\sum_{j=1}^m w_j^2} \left\{ \bar{y}_w - \sum_{j=1}^m w_j \hat{\xi}_j^{AEB} \right\}, \quad (4.2)$$

which is here called the constrained approximate Bayes estimator.

To evaluate the uncertainty of δ_i^{CAB} , we derive a second-order unbiased estimator of the MSE. The MSE of δ_i^{CAB} is decomposed as

$$\begin{aligned} E[(\delta_i^{CAB} - \xi_i)^2] &= E[(\hat{\xi}_i^{AEB} - \xi_i)^2] \\ &\quad + \frac{w_i^2}{(\sum_{j=1}^m w_j^2)^2} E\left[(\bar{y}_w - \sum_{j=1}^m w_j \hat{\xi}_j^{AEB})^2\right] + 2 \frac{w_i}{\sum_{j=1}^m w_j^2} J, \end{aligned}$$

where

$$J = E\left[(\hat{\xi}_i^{AEB} - \xi_i) \left(\bar{y}_w - \sum_{j=1}^m w_j \hat{\xi}_j^{AEB} \right)\right].$$

The second-order unbiased estimator of the first term $E[(\hat{\xi}_i^{AEB} - \xi_i)^2]$ is given in Proposition 3.2. Clearly, an exact unbiased estimator of $E[(\bar{y}_w - \sum_{j=1}^m w_j \hat{\xi}_j^{AEB})^2]$ is $(\bar{y}_w - \sum_{j=1}^m w_j \hat{\xi}_j^{AEB})^2$. In Theorem 4.1 given below, we verify that $J = O(1)$. Then, we can estimate J based on the bootstrap sample given in (3.7) as

$$J^* = E^*\left[(\hat{\xi}_i^{AEB*} - \xi_i^*) \left(\bar{y}_w^* - \sum_{j=1}^m w_j \hat{\xi}_j^{AEB*} \right)\right],$$

which satisfies that $E[J^*] = J + o(1)$.

Theorem 4.1 *Assume the condition (A) and $n_i + \alpha > 4$ for $i = 1, \dots, m$. Also assume that $\sum_{j=1}^m w_j^2/m$ converges to a non-zero constant. Then, $J = O(1)$ and a second-order unbiased estimator of the MSE of δ_i^{CAB} is*

$$mse(\delta_i^{CAB}) = mse(\hat{\xi}_i^{AEB}) + \frac{w_i^2}{(\sum_{j=1}^m w_j^2)^2} \left[(\bar{y}_w - \sum_{j=1}^m w_j \hat{\xi}_j^{AEB})^2 \right] + 2 \frac{w_i}{\sum_{j=1}^m w_j^2} J^*, \quad (4.3)$$

where $mse(\hat{\xi}_i^{AEB})$ is given in (3.9). That is, $E[mse(\delta_i^{CAB})] = E[(\delta_i^{CAB} - \xi_i)^2] + o(m^{-1})$.

The proof of Theorem 4.1 is given in the Appendix.

5 Numerical and Empirical Studies

In this section, we investigate performances of the procedures suggested in the previous sections through numerical and empirical studies.

5.1 Simulation study

Here we investigate finite sample performances of the estimators in the Fay-Herriot random dispersion (FHRD) model and the second-order unbiased estimators for the unconditional MSEs by the Monte Carlo simulation. Comparing the performances of the approximated Bayes estimator ξ_i^{AB} given in (2.8) with those of the Bayes estimator ξ_i^B given in (2.4), we check goodness of the approximation we applied.

We conduct simulation experiments as we specified true model, so simulation data is generated by FHRD model (2.1). Throughout the simulations, the true value of β and γ are $\beta = 10$ and $\gamma = 1$. For each of m, τ^2, α , we examined two cases; $m = 30$ or 60 , $\tau^2 = 1$ or 4 and $\alpha = 1$ or 4 . For simplicity, we set $z_i = 1$, $p = 1$ and $n_i = 10$ for all areas and cases. Thus, there are eight cases of simulations for the variety of m, α and τ^2 .

We first compute numerical values of MSE of the Bayes and approximate Bayes estimators ξ_i^B and ξ_i^{AB} with

$$\text{MSE}_i = \frac{1}{K} \sum_{k=1}^K \left(\hat{\xi}_i^{(k)} - \xi_i^{(k)} \right)^2$$

for $K = 5,000$ where $\hat{\xi}_i^{(k)}$ and $\xi_i^{(k)}$ are the estimator and the true value of ξ_i in the k -th replication for $k = 1, \dots, K$. To investigate the loss which arises from approximation (2.7), we compare the approximated Bayes estimator ξ_i^{AB} with the Bayes estimator ξ_i^B in terms of biases and true MSEs under known model parameters. The results of the simulation are reported in Table 1.

Table 1: Biases and square roots of MSE of ξ_i^{AB} and ξ_i^B in the FHRD model under known model parameters

Estimator	α, τ^2	<i>Bias</i> (m=30)	<i>SRMSE</i> (m=30)	<i>Bias</i> (m=60)	<i>SRMSE</i> (m=60)
ξ_i^B	1, 1	0.002	0.818	-0.002	0.822
	1, 4	0.003	1.347	-0.005	1.346
	4, 1	-0.005	0.528	0.000	0.528
	4, 4	-0.005	0.629	-0.002	0.628
ξ_i^{AB}	1, 1	0.002	0.824	-0.002	0.828
	1, 4	0.003	1.355	-0.005	1.355
	4, 1	-0.006	0.530	0.001	0.530
	4, 4	-0.005	0.631	-0.002	0.629

It is observed from Table 1 that the difference between ξ_i^B and ξ_i^{AB} is tiny. Even though ξ_i^{AB} is dominated by ξ_i^B as expected, the difference of the biases is little except the case $(\alpha, \tau^2) = (4, 1)$. Moreover, the largest difference between the two SRMSEs is

0.009 for $m = 60$ and $(\alpha, \tau^2) = (1, 4)$. Thus, the approximation little affects the bias and MSE.

We next investigate finite sample performances of the estimators for the model parameters. In particular, it is worth remarking the estimation of α . Initially, we used the maximum likelihood estimator of α from the marginal likelihood as Maiti et al. (2014). The MLE can be obtained by solving the equation based on the digamma functions. However, the numerical solutions for the MLE yield large variability. To avoid such an instable performance of the MLE, in this paper, we suggest the new consistent estimator given in (2.19). The performances of the suggested estimators for β , τ^2 , α and γ are reported in Table 2, where means and standard deviations via simulation with 1,000 replications are given. Table 2 shows that the estimator of β is almost unbiased and has small standard deviation. For other estimators, both biases and standard deviations are moderated as m increases. Espetially, the suggested estimator (2.19) of α provides stable estimates and a good performance.

Table 2: Results of the estimation for model parameters β , τ^2 , α and γ in the FHRD model for $\beta = 10$ and $\gamma = 1$ (Standard deviations are shown in the parentheses)

m	α, τ^2	$\hat{\beta}$	$\hat{\tau}^2$	$\hat{\alpha}$	$\hat{\gamma}$
$m = 30$	1, 1	10.001(0.331)	0.912(0.658)	1.041(0.160)	1.092(0.262)
	1, 4	9.997(0.495)	3.658(1.834)	1.038(0.157)	1.092(0.269)
	4, 1	10.000(0.217)	0.950(0.361)	4.067(0.538)	1.013(0.085)
	4, 4	10.006(0.384)	3.876(1.235)	4.063(0.543)	1.015(0.085)
$m = 60$	1, 1	9.999(0.226)	0.944(0.483)	1.135(0.189)	1.203(0.346)
	1, 4	10.006(0.345)	3.883(1.321)	1.018(0.114)	1.036(0.167)
	4, 1	10.000(0.148)	0.977(0.254)	4.029(0.377)	1.006(0.061)
	4, 4	9.998(0.268)	3.928(0.880)	4.036(0.383)	1.005(0.062)

Finally, we compare the second-order unbiased estimator of the MSE of $\hat{\xi}_i^{AEB}$ with the true MSE. Concerning the MSE, the true value is calculated via simulation with $R = 5,000$ replications as 3.4. Then, the mean values of the estimator for the MSE and their Percentage Relative Bias (RB) are calculated based on $T = 1,000$ simulation runs with each 1,000 bootstrap samples, where RB is defined by

$$RB_i = 100 \frac{T^{-1} \sum_{t=1}^T \widehat{MSE}_i^{(t)} - MSE_i}{MSE_i},$$

for the MSE estimate $\widehat{MSE}_i^{(t)}$ in the t -th replication for $t = 1, \dots, T$. Note that both true MSE and its estimator are calculated based on our estimates of the model parameters.

Table 3 reports values of the true MSE, the second-order unbiased estimator \widehat{MSE} and the corresponding Percentage Relative Biases. It is observed that the MSE estimates \widehat{MSE} are close to the true values of the MSE, and their relative bias are small for both $m = 30$ and 60. Thus, the second-order unbiased estimator of the MSE performs well as an estimator of the MSE of $\hat{\xi}_i^{AEB}$ in the FHRD model.

Table 3: Relative biases of the MSE estimator for $\hat{\xi}_I^{AEB}$ in the FHRD model

Size	α, τ^2	MSE	\widehat{MSE}	RB(%)
$m = 30$	1, 1	0.996	0.952	-4.379
	1, 4	2.22	2.243	1.042
	4, 1	0.312	0.323	3.580
	4, 4	0.416	0.414	-0.443
$m = 60$	1, 1	0.835	0.833	-0.233
	1, 4	2.026	2.011	-0.719
	4, 1	0.294	0.298	1.284
	4, 4	0.399	0.405	1.424

5.2 Illustrative examples

We apply the approximated empirical Bayes estimator and the estimator of the MSE to the data in the Survey of Family Income and Expenditure (SFIE) in Japan.

In this study, we use the data of the spending items ‘Education’ and ‘Health’ in the survey in 2014. For the spending item ‘Education’, the annual average spending (scaled by 1,000 Yen) at each capital city of 47 prefectures in Japan is denoted by y_i for $i = 1, \dots, 47$, and each variance V_i is calculated based on data of the spending ‘Education’ at the same city in the past consecutive eight years. Although the annual average spendings in SFIE are reported every year, the sample sizes are around 50 for most prefectures. We apply the same manner to the spending item ‘Health’ to create y_i and V_i . The data of the item ‘Education’ have high variability, but those of the item ‘Health’ have relatively lower one.

In addition to the SFIE data, we can use data in the National Survey of Family Income and Expenditure (NSFIE) for 47 prefectures. Since NSFIE is based on much larger sample than SFIE, the annual average spendings in NSFIE are more reliable, but this survey has been implemented every five years. In this study, we use the data of the spending items ‘Education’ and ‘Health’ of NSFIE in 2009 as covariates z_i for $i = 1, \dots, 47$. Thus, we apply the FHRD model (2.1) to these examples, where $\mathbf{z}_i^T \boldsymbol{\beta} = \beta_0 + z_i \beta_1$.

We calculated the predicted values of $\hat{\xi}_i^{AEB}$ and $\hat{\xi}_i^{EB}$ and the estimates of the MSE of $\hat{\xi}_i^{AEB}$ with the estimates V_i . These values in seven prefectures around Tokyo are reported

in Tables 4 and 5 for the ‘Education’ and ‘Health’ data. For the ‘Education’ data, the estimates of the model parameters are $\hat{\beta}_0 = 15.711$, $\hat{\beta}_1 = 0.140$, $\hat{\tau}^2 = 12.069$, $\hat{\alpha} = 2.050$ and $\hat{\gamma} = 2.764$. For the ‘Health’ data, the estimates of the model parameters are $\hat{\beta}_0 = 8.819$, $\hat{\beta}_1 = 0.192$, $\hat{\tau}^2 = 5.497$, $\hat{\alpha} = 9.502$ and $\hat{\gamma} = 2.109$. As seen from the tables, the ‘Education’ data have more variability than the ‘Health’ data. The approximated empirical Bayes estimator $\hat{\xi}_i^{AEB}$ returns almost similar values as the empirical Bayes estimator $\hat{\xi}_i^{EB}$ does. Both estimators do not shrink y_i so much. It is also from Table 4 that $\hat{\xi}_i^{AEB}$ and $\hat{\xi}_i^{EB}$ shrink y_i more toward $\hat{\beta}_0 + z_i \hat{\beta}_1$ when the values of V_i are larger. The MSE estimates of $\hat{\xi}_i^{AEB}$ give large values for large V_i 's.

Table 4: Predicted values and the MSE estimates for the ‘Education’ data

Prefecture	V_i	y_i	$z_i^T \hat{\beta}$	$\hat{\xi}_i^{AEB}$	$\hat{\xi}_i^{EB}$	\widehat{MSE}_{AEB}
Ibaraki	4.210	21.972	17.873	21.768	21.851	1.098
Tochigi	4.974	21.883	18.102	21.675	21.768	1.157
Gunma	11.157	14.115	17.933	14.475	14.287	1.772
Saitama	72.622	32.608	19.309	27.805	29.064	5.868
Chiba	26.419	21.554	18.751	21.050	21.274	3.144
Tokyo	13.091	22.037	19.337	21.750	21.915	2.084
Kanagawa	16.266	22.321	18.494	21.843	22.106	2.260

Table 5: Predicted values and the MSE estimates for the ‘Health’ data

Prefecture	V_i	y_i	$z_i^T \hat{\beta}$	$\hat{\xi}_i^{AEB}$	$\hat{\xi}_i^{EB}$	\widehat{MSE}_{AEB}
Ibaraki	1.160	10.351	10.946	10.369	10.410	0.211
Tochigi	3.964	11.759	11.080	11.720	11.572	0.369
Gunma	3.444	8.737	10.307	8.818	9.139	0.349
Saitama	0.920	11.133	11.316	11.138	11.146	0.194
Chiba	3.720	12.808	11.150	12.718	12.388	0.358
Tokyo	1.161	13.803	10.959	13.714	13.477	0.208
Kanagawa	0.479	14.496	11.088	14.411	14.293	0.163

6 Concluding Remarks

In the Fay-Herriot random dispersion (FHRD) model, we have derived the approximated empirical Bayes (AEB) estimator with the closed form and provided the second-order

unbiased estimator of the MSE of the AEB estimator via the single parametric bootstrap method. Through various simulation experiments and empirical studies, it has been shown that the difference between the AEB estimator and the empirical Bayes estimator given in Maiti, *et al.* (2014) is small. This means that the AEB estimator is useful irrespective of validity of the approximation. It has been observed that the estimators suggested in this paper for the model parameters have good performances. Especially, our estimator of α is described in the closed form, and it performs well.

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A Appendix

We here give proofs of Proposition 2.1 and Theorems 3.1 and 4.1.

[1] **Proof of Proposition 2.1.** It follows from (2.5) and (2.6) that

$$E\left[\frac{1}{\tau^2\eta_i + 1} \mid y_i, V_i\right] = \frac{\int_0^\infty (\tau^2\eta_i + 1)^{-1} D_i(\eta_i) g_i^*(\eta_i) d\eta_i}{\int_0^\infty D_i(\eta_i) g_i^*(\eta_i) d\eta_i},$$

where

$$g_i^*(\eta_i) = \eta_i^{(n_i+1+\alpha)/2-1} \exp[-\eta_i(V_i + \gamma)/2],$$

$$D_i(\eta_i) = \frac{1}{\sqrt{\tau^2\eta_i + 1}} \exp\left[-\frac{\eta_i}{2(\tau^2\eta_i + 1)}(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2\right].$$

It is noted that $D_i(\eta_i)$ is a decreasing function of η_i . Then, we first show that

$$\frac{\int_0^\infty (\tau^2\eta_i + 1)^{-1} D_i(\eta_i) g_i^*(\eta_i) d\eta_i}{\int_0^\infty D_i(\eta_i) g_i^*(\eta_i) d\eta_i} \geq \frac{\int_0^\infty (\tau^2\eta_i + 1)^{-1} g_i^*(\eta_i) d\eta_i}{\int_0^\infty g_i^*(\eta_i) d\eta_i}. \quad (\text{A.1})$$

This inequality is equivalent to

$$E_*[(\tau^2\eta_i + 1)^{-1} D_i(\eta_i)] \geq E_*[(\tau^2\eta_i + 1)^{-1}] E_*[D_i(\eta_i)], \quad (\text{A.2})$$

where $E_*[\cdot]$ is the expectation with respect to the pdf $C_i g_i^*(\eta_i)$ for some constant C_i . The inequality (A.2) is equivalent to

$$\begin{aligned} & Cov_*((\tau^2\eta_i + 1)^{-1}, D_i(\eta_i)) \\ &= E_*\left[\left\{(\tau^2\eta_i + 1)^{-1} - E_*[(\tau^2\eta_i + 1)^{-1}]\right\} \left\{D_i(\eta_i) - E_*[D_i(\eta_i)]\right\}\right] \geq 0, \end{aligned}$$

which holds since both $(\tau^2\eta_i + 1)^{-1}$ and $D_i(\eta_i)$ are decreasing in η_i . Thus, one gets the inequality (A.1).

Since $(\tau^2\eta_i + 1)^{-1}$ is a convex function of η_i , the Jensen inequality is applied to show the inequality

$$\frac{\int_0^\infty (\tau^2\eta_i + 1)^{-1} g_i^*(\eta_i) d\eta_i}{\int_0^\infty g_i^*(\eta_i) d\eta_i} \geq \left(\tau^2 \frac{\int_0^\infty \eta_i g_i^*(\eta_i) d\eta_i}{\int_0^\infty g_i^*(\eta_i) d\eta_i} + 1 \right)^{-1}.$$

Noting that $g_i^*(\eta_i)$ is proportional to the pdf of $Ga((n_i + 1 + \alpha)/2, 2/(V_i + \gamma))$, we can see that $\int_0^\infty \eta_i g_i^*(\eta_i) d\eta_i / \int_0^\infty g_i^*(\eta_i) d\eta_i = (n_i + \alpha + 1)/(V_i + \gamma)$. Namely,

$$\frac{\int_0^\infty (\tau^2\eta_i + 1)^{-1} g_i^*(\eta_i) d\eta_i}{\int_0^\infty g_i^*(\eta_i) d\eta_i} \geq \frac{1}{\tau^2(n_i + \alpha + 1)/(V_i + \gamma) + 1}. \quad (\text{A.3})$$

Combining (A.1) and (A.3) proves Proposition 2.1.

[2] Proof of Theorem 3.1. For notational simplicity, let $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^T = (\tau^2, \alpha, \gamma)^T$ and $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)^T = (\hat{\tau}^2, \hat{\alpha}, \hat{\gamma})^T$. Since $\hat{\boldsymbol{\beta}}$ is expressed as $\hat{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}}(\hat{\boldsymbol{\theta}})$, the Taylor series expansion gives that

$$\begin{aligned} \hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = & \tilde{\boldsymbol{\beta}} - \boldsymbol{\beta} + \sum_{a=1}^3 \left\{ \left(\sum_{j=1}^m (1 - B_j) \mathbf{z}_j \mathbf{z}_j^T \right)^{-1} \left(\sum_{j=1}^m \frac{\partial B_j}{\partial \theta_a} \mathbf{z}_j \mathbf{z}_j^T \right) \tilde{\boldsymbol{\beta}} \right. \\ & \left. - \left(\sum_{j=1}^m (1 - B_j) \mathbf{z}_j \mathbf{z}_j^T \right)^{-1} \sum_{j=1}^m \frac{\partial B_j}{\partial \theta_a} \mathbf{z}_j \mathbf{z}_j^T \boldsymbol{\beta}^* \right\} (\hat{\theta}_a - \theta_a) + o_p(m^{-1}), \end{aligned} \quad (\text{A.4})$$

where $\boldsymbol{\beta}^* = \left\{ \sum_{j=1}^m (\partial B_j / \partial \theta_a) \mathbf{z}_j \mathbf{z}_j^T \right\}^{-1} \sum_{j=1}^m (\partial B_j / \partial \theta_a) \mathbf{z}_j y_j$. Since $E[(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 | V_i] = (V_i + \gamma)/(n_i + \alpha - 2) + \tau^2$ from (2.13), it is observed that

$$\begin{aligned} E[(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})^T | V_i] = & E \left[\left(\sum_{j=1}^m (1 - B_j) \mathbf{z}_j \mathbf{z}_j^T \right)^{-1} \left(\sum_{j=1}^m \mathbf{z}_j \mathbf{z}_j^T (1 - B_j)^2 \left\{ \frac{V_j + \gamma}{n_j + \alpha - 2} + \tau^2 \right\} \right) \right. \\ & \left. \times \left(\sum_{j=1}^m (1 - B_j) \mathbf{z}_j \mathbf{z}_j^T \right)^{-1} | V_i \right], \end{aligned}$$

which implies that $E[(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})^T] = O(m^{-1})$, and $\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta} = O_p(m^{-1/2})$ under the condition (A) for $n_j + \alpha > 2$. Also, it is seen that

$$\begin{aligned} E[(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})^T | y_i, V_i] = & E[(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})^T] \\ & + E \left[\left(\sum_{j=1}^m (1 - B_j) \mathbf{z}_j \mathbf{z}_j^T \right)^{-1} \mathbf{z}_i \mathbf{z}_i^T (1 - B_i)^2 \left((y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 - \left\{ \frac{V_j + \gamma}{n_j + \alpha - 2} + \tau^2 \right\} \right) \right. \\ & \left. \times \left(\sum_{j=1}^m (1 - B_j) \mathbf{z}_j \mathbf{z}_j^T \right)^{-1} | y_i, V_i \right], \end{aligned} \quad (\text{A.5})$$

which shows that $E[(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})^T | y_i, V_i] = E[(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})^T] + O_p(m^{-2})$ and $\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta} | y_i, V_i = O_p(m^{-1/2})$. Clearly, $E[\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}] = \mathbf{0}$, and it is observed that

$$\begin{aligned} E[\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta} | y_i, V_i] &= E\left[\left\{\sum_{j=1}^m (1 - B_j) \mathbf{z}_j \mathbf{z}_j^T\right\}^{-1} \sum_{j=1}^m (1 - B_j) \mathbf{z}_j (y_j - \mathbf{z}_j^T \boldsymbol{\beta}) | y_i, V_i\right] \\ &= E\left[\left\{\sum_{j=1}^m (1 - B_j) \mathbf{z}_j \mathbf{z}_j^T\right\}^{-1} | y_i, V_i\right] (1 - B_i) \mathbf{z}_i (y_i - \mathbf{z}_i^T \boldsymbol{\beta}), \end{aligned}$$

which means that $E[\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta} | y_i, V_i] = O_p(m^{-1})$. Similarly, we can show that $E[\boldsymbol{\beta}^* - \boldsymbol{\beta} | y_i, V_i] = O_p(m^{-1})$ and $\boldsymbol{\beta}^* - \boldsymbol{\beta} | y_i, V_i = O_p(m^{-1/2})$. Thus, from (A.4), we can verify that $E[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})^T] = O(m^{-1})$, $E[\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} | y_i, V_i] = +o_p(m^{-1})$ and $E[(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta})^T | y_i, V_i] = E[(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta})^T] + o_p(m^{-1})$, provided $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)^T$ satisfy the results given in Theorem 3.1.

We now prove the results concerning $\hat{\boldsymbol{\theta}}$ in Theorem 3.1. Let $\mathbf{F}(\boldsymbol{\theta}) = (F_1(\boldsymbol{\theta}), F_2(\boldsymbol{\theta}), F_3(\boldsymbol{\theta}))^T$, where

$$\begin{aligned} F_1(\boldsymbol{\theta}) &= \frac{1}{m} \sum_{j=1}^m \left\{ \frac{\alpha/\gamma}{n_j + \alpha} \tau^2 + \frac{1}{n_j + \alpha - 2} - \frac{(y_j - \mathbf{z}_j^T \hat{\boldsymbol{\beta}}_{OLS})^2}{V_j + \gamma} \right\}, \\ F_2(\boldsymbol{\theta}) &= \frac{1}{m} \sum_{j=1}^m \left\{ \alpha^2 \frac{V_j}{V_j + \gamma} \log(V_j + \gamma) + \alpha n_j \frac{V_j - \gamma}{V_j + \gamma} \log(V_j + \gamma) - \frac{n_j^2 \gamma}{V_j + \gamma} \log(V_j + \gamma) - 2n_j \right\}, \\ F_3(\boldsymbol{\theta}) &= \frac{1}{m} \sum_{j=1}^m \left\{ \frac{V_j}{V_j + \gamma} - \frac{n_j}{n_j + \alpha} \right\}. \end{aligned}$$

Since $\mathbf{F}(\hat{\boldsymbol{\theta}}) = \mathbf{0}$, the consistency of $\hat{\boldsymbol{\theta}}$ follows from the Cramer method explained in Jiang (2010). It is noted that for $a = 1, 2, 3$,

$$F_a(\hat{\boldsymbol{\theta}}) = F_a(\boldsymbol{\theta}) + \left(\frac{\partial F_a(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T} \right) (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + \frac{1}{2} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T \frac{\partial^2 F_a}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) + o_p(\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^2),$$

which yields

$$\begin{aligned} \hat{\boldsymbol{\theta}} - \boldsymbol{\theta} &= - \left(\frac{\partial \mathbf{F}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T} \right)^{-1} \mathbf{F}(\boldsymbol{\theta}) - \frac{1}{2} \left(\frac{\partial \mathbf{F}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T} \right)^{-1} \mathbf{Col}_a \left((\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T \frac{\partial^2 F_a}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^T} (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \right) \\ &\quad + o_p(\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^2), \end{aligned} \tag{A.6}$$

where $\mathbf{Col}_a(x_a) = (x_1, x_2, x_3)^T$ and $\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^2 = (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$. Hence, it is sufficient to show that $E[\{F_a(\boldsymbol{\theta})\}^2] = O(m^{-1})$, $E[\{F_a(\boldsymbol{\theta})\}^2 | y_i, V_i] = O_p(m^{-1})$, $E[F_a(\boldsymbol{\theta})] = O_p(m^{-1})$ and $E[F_a(\boldsymbol{\theta}) | y_i, V_i] = O_p(m^{-1})$ for each a , and $\partial \mathbf{F}(\boldsymbol{\theta}) / \partial \boldsymbol{\theta}^T$ converges to a positive definite matrix.

Concerning $F_1(\boldsymbol{\theta})$, note that

$$F_1(\boldsymbol{\theta}) = F_1^*(\boldsymbol{\beta}, \boldsymbol{\theta}) + \frac{2}{m} \sum_{j=1}^m \frac{y_j - \mathbf{z}_j^T \boldsymbol{\beta}}{V_j + \gamma} \mathbf{z}_j^T (\widehat{\boldsymbol{\beta}}_{OLS} - \boldsymbol{\beta}) + O_p(m^{-1}),$$

where

$$F_1^*(\boldsymbol{\beta}, \boldsymbol{\theta}) = \frac{1}{m} \sum_{j=1}^m \left\{ \frac{\alpha/\gamma}{n_j + \alpha} \tau^2 + \frac{1}{n_j + \alpha - 2} - \frac{(y_j - \mathbf{z}_j^T \boldsymbol{\beta})^2}{V_j + \gamma} \right\}.$$

It can be verified that $E[\{F_1(\boldsymbol{\theta})\}^2] = O(m^{-1})$, $E[\{F_1(\boldsymbol{\theta})\}^2 | y_i, V_i] = O_p(m^{-1})$, $E[F_1(\boldsymbol{\theta})] = O_p(m^{-1})$ and $E[F_1(\boldsymbol{\theta}) | y_i, V_i] = O_p(m^{-1})$ if there exists $E[\{(y_j - \mathbf{z}_j^T \boldsymbol{\beta})^4 / (V_j + \gamma)^2\}]$. Similarly, the corresponding properties of the moments for $F_2(\boldsymbol{\theta})$ and $F_3(\boldsymbol{\theta})$ can be demonstrated if there exists $E[\{\log(V_j + \gamma)\}^2]$. Thus, we need to check these moments. It is noted that

$$\begin{aligned} E[\{(y_j - \mathbf{z}_j^T \boldsymbol{\beta})^4 | V_j\}] &= E[(y_j - \xi_j)^4 + 6(y_j - \xi_j)^2(\xi_j - \mathbf{z}_j^T \boldsymbol{\beta})^2 + (\xi_j - \mathbf{z}_j^T \boldsymbol{\beta})^4 | V_j] \\ &= E\left[\frac{3}{\eta_j^2} + \frac{6\tau^2}{\eta_j} + 3\tau^4 | V_j\right] \\ &= \frac{3(V_j + \gamma)^2}{(n_j + \alpha - 2)(n_j + \alpha - 4)} + \frac{6\tau^2(V_j + \gamma)}{n_j + \alpha - 2} + 3\tau^4, \end{aligned}$$

so that $E[\{(y_j - \mathbf{z}_j^T \boldsymbol{\beta})^4 / (V_j + \gamma)^2\}]$ is finite if $n_j + \alpha > 4$. To investigate the existence of $E[\{\log(V_j + \gamma)\}^2]$, we calculate both sides of $4E[-(\partial^2 / \partial \alpha^2) \log f(V_j)] = 4E[\{(\partial / \partial \alpha) \log f(V_j)\}^2]$. The RHS is equal to $E[\{\psi((n_j + \alpha)/2) - \psi(\alpha/2) + \log \gamma - \log(V_j + \gamma)\}^2] = \{\psi((n_j + \alpha)/2) - \psi(\alpha/2) + \log \gamma\}^2 - E[\{\log(V_j + \gamma)\}^2]$. On the other hand, the LHS is $-\psi'((n_j + \alpha)/2) + \psi'(\alpha/2)$. Hence,

$$E[\{\log(V_j + \gamma)\}^2] = \{\psi((n_j + \alpha)/2) - \psi(\alpha/2) + \log \gamma\}^2 + \psi'((n_j + \alpha)/2) - \psi'(\alpha/2),$$

which is finite for $\alpha > 0$.

Finally, we show that every entry of the matrix $\partial \mathbf{F}(\boldsymbol{\theta}) / \partial \boldsymbol{\theta}^T$ converges in probability. For notational simplicity, let $F_{a(i)}(\boldsymbol{\theta}) = \partial F_a(\boldsymbol{\theta}) / \partial \theta_i$ for $a = 1, 2, 3$ and $i = 1, 2, 3$. Then, for $F_{1(i)}$, we have $F_{1(1)} = m^{-1} \sum_j \alpha / \{\gamma(n_j + \alpha)\}$, and

$$F_{1(2)} = \frac{1}{m} \sum_j \left\{ \frac{n_j \tau^2 / \gamma}{(n_j + \alpha)^2} - \frac{1}{(n_j + \alpha - 2)^2} \right\}, F_{1(3)} = \frac{1}{m} \sum_j \left\{ -\frac{\tau^2 / \gamma^2}{n_j + \alpha} + \frac{(y_j - \mathbf{z}_j^T \widehat{\boldsymbol{\beta}}_{OLS})^2}{(V_j + \gamma)^2} \right\}.$$

For $F_{2(i)}$, we have $F_{2(1)} = 0$, and

$$\begin{aligned} F_{2(2)} &= \frac{1}{m} \sum_j \frac{\log(V_j + \gamma)}{V_j + \gamma} \{2\alpha V_j + n_j(V_j - \gamma)\}, \\ F_{2(3)} &= -\frac{1}{m} \sum_j \frac{V_j \log(V_j + \gamma)}{(V_j + \gamma)^2} (\alpha + n_j)^2 + \frac{1}{m} \sum_j \frac{\alpha + n_j}{(V_j + \gamma)^2} \{\alpha V_j - n_j \gamma\}. \end{aligned}$$

For $F_{3(i)}$, we have $F_{3(1)} = 0$, and

$$F_{3(2)} = \frac{1}{m} \sum_j \frac{n_j}{(n_j + \alpha)^2}, \quad F_{3(3)} = -\frac{1}{m} \sum_j \frac{V_j}{(V_j + \gamma)^2}.$$

Thus, it can be seen that these converge to the limiting values of their expectations if there exist the moments $E[\{(y_j - \mathbf{z}_j^T \boldsymbol{\beta})^4 / (V_j + \gamma)^2\}]$ and $E[\{\log(V_j + \gamma)\}^2]$. are finite. Therefore, the proof of Theorem 3.1 is complete.

[3] Proof of Theorem 4.1. It is noted from (2.4) that $\xi_i^B = E[\xi_i | y_i, V_i] = \mathbf{z}_i^T \boldsymbol{\beta} + (1 - E_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta})$ for $E_i = E[(\tau^2 \eta_i + 1)^{-1} | y_i, V_i]$. Then, J can be rewritten as

$$\begin{aligned} J &= E \left[\left(\widehat{\xi}_i^{AEB} - \xi_i^B \right) \left(\bar{y}_w - \sum_{j=1}^m w_j \widehat{\xi}_j^{AEB} \right) \right] \\ &= E \left[(B_i - E_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta}) \sum_{j=1}^m w_j B_j (y_j - \mathbf{z}_j^T \boldsymbol{\beta}) \right] \\ &\quad + E \left[\left\{ \widehat{B}_i \mathbf{z}_i^T (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) - (\widehat{B}_i - B_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta}) \right\} \sum_{j=1}^m w_j \widehat{B}_j (y_j - \mathbf{z}_j^T \widehat{\boldsymbol{\beta}}) \right] \\ &\quad + E \left[(B_i - E_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta}) \sum_{j=1}^m w_j \left\{ (\widehat{B}_j - B_j)(y_j - \mathbf{z}_j^T \widehat{\boldsymbol{\beta}}) - B_j \mathbf{z}_j^T (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \right\} \right] \\ &= J_1 + J_2 + J_3. \quad (\text{say}) \end{aligned}$$

Noting that E_i is a function of V_i and $(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2$, we can see that

$$E[(B_i - E_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta})(y_j - \mathbf{z}_j^T \boldsymbol{\beta}) | V_i, V_j] = \begin{cases} (B_i - E_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Thus, J_1 is evaluated as

$$\begin{aligned} J_1 &= E \left[w_i B_i (B_i - E_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 + (B_i - E_i) \sum_{j \neq i} w_j B_j (y_i - \mathbf{z}_i^T \boldsymbol{\beta})(y_j - \mathbf{z}_j^T \boldsymbol{\beta}) \right] \\ &= E[w_i B_i (B_i - E_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2], \end{aligned}$$

which is of order $O(1)$. Noting that $\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta} = O_p(m^{-1/2})$ and $\widehat{B}_i - B_i = O_p(m^{-1/2})$, we can demonstrate that

$$\begin{aligned} J_2 &= E \left[\left\{ B_i \mathbf{z}_i^T (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) - (\widehat{B}_i - B_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta}) \right\} \sum_{j=1}^m w_j B_j (y_j - \mathbf{z}_j^T \boldsymbol{\beta}) \right] + o(1) \\ J_3 &= E \left[(B_i - E_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta}) \sum_{j=1}^m w_j \left\{ (\widehat{B}_j - B_j)(y_j - \mathbf{z}_j^T \boldsymbol{\beta}) - B_j \mathbf{z}_j^T (\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \right\} \right] + o(1). \end{aligned}$$

It is easily seen that

$$E \left[\left\{ \sum_{j=1}^m w_j B_j (y_j - \mathbf{z}_j^T \boldsymbol{\beta}) \right\}^2 \mid V_1, \dots, V_m \right] = E \left[\sum_{j=1}^m w_j^2 B_j^2 (y_j - \mathbf{z}_j^T \boldsymbol{\beta})^2 \mid V_1, \dots, V_m \right],$$

which implies that $\sum_{j=1}^m w_j B_j (y_j - \mathbf{z}_j^T \boldsymbol{\beta}) = O_p(m^{-1/2})$. Thus, one gets that $J_2 = O(1)$.

Concerning J_3 , it is noted that $\hat{\tau}^2$ given in (2.15) is approximated as

$$\hat{\tau}^2 = \left(\sum_{i=1}^m \frac{\alpha/\gamma}{n_i + \alpha} \right)^{-1} \sum_{i=1}^m \left\{ \frac{(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2}{V_i + \gamma} - \frac{1}{n_i + \alpha - 2} \right\} + O_p(m^{-1/2}),$$

so that we can regard \hat{B}_i as a function of $(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2$ and V_i , $i = 1, \dots, m$. Hence,

$$E \left[(B_i - E_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta}) \sum_{j=1}^m w_j (\hat{B}_j - B_j)(y_j - \mathbf{z}_j^T \boldsymbol{\beta}) \right] = E[(B_i - E_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 w_i (\hat{B}_i - B_i)],$$

which is of $O(m^{-1/2})$. Since $\hat{\boldsymbol{\beta}} - \tilde{\boldsymbol{\beta}} = O_p(m^{-1/2})$, it is seen that

$$\begin{aligned} & E \left[(B_i - E_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta}) \sum_{j=1}^m w_j B_j \mathbf{z}_j^T (\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}) \mid V_1, \dots, V_m \right] \\ &= E \left[(B_i - E_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta}) \sum_{j=1}^m w_j B_j \mathbf{z}_j^T \left(\sum_{a=1}^m (1 - B_a) \mathbf{z}_a \mathbf{z}_a^T \right)^{-1} \right. \\ & \quad \left. \times \sum_{a=1}^m (1 - B_a) \mathbf{z}_a (y_a - \mathbf{z}_a^T \boldsymbol{\beta}) \mid V_1, \dots, V_m \right] \\ &= E \left[(B_i - E_i)(y_i - \mathbf{z}_i^T \boldsymbol{\beta})^2 \sum_{j=1}^m w_j B_j \mathbf{z}_j^T \left(\sum_{a=1}^m (1 - B_a) \mathbf{z}_a \mathbf{z}_a^T \right)^{-1} (1 - B_i) \mathbf{z}_i \mid V_1, \dots, V_m \right], \end{aligned}$$

which is of order $O_p(1)$. Hence, it is concluded that $J_3 = O(1)$. Therefore, the proof of Theorem 4.1 is complete.

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