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Interlinkage and Generous Tit-for-Tat Strategy

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Abstract

We investigate an infinitely repeated prisoners' dilemma with imperfect monitoring and projects the possibility that the interlinkage of the players' distinct activities enhances implicit collusion. We show a necessary and sufficient condition for the existence of generous tit-for-tat Nash equilibrium. Such an equilibrium, if it exists, is unique. This equilibrium achieves approximate efficiency when monitoring is almost perfect, where the discount factors are fixed.

Keywords: Interlinkage, Repeated Games, Imperfect Monitoring, Generous Tit-For-Tat, Approximate Efficiency

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1. Introduction

It is a generally accepted view in economics that when an economic agent lacks means such as collaterals to incentivize his partner, he prefers interlinking the partner's activity with his own, both of which are originally unrelated; the agent makes his activity contingent on the partner's performance to establish an implicit collusion. This paper models this interlinkage as an infinitely repeated prisoners' dilemma, where the players' payoff functions have additively separable forms, thereby implying that there is no synergy between their activities, except in that they both happen to be carried out by the same pair of economic agents. We assume that imperfect monitoring exists: each player cannot directly observe the other player's action, but imperfectly monitors it through a random signal. In this paper, we demonstrate the possibility that the collusive behavior of the players is sustained by Nash equilibrium.

We focus on Nash equilibria that induce the players to follow *generous tit-for-tat* strategies; each player starts selecting a cooperative action. He selects it if he detects a good signal for the other player's action, whereas he selects a defective action with a positive probability if he detects a bad signal. This strategy may be so generous that even after observing a bad signal, the player selects a cooperative action with a positive probability. We characterize a necessary and sufficient condition for the existence of a generous tit-for-tat Nash equilibrium, which, if it exists, is unique. This equilibrium approximately induces efficiency when monitoring is almost perfect.

The generous tit-for-tat strategy has been studied in biology and is known to survive in computer-based tournaments. See Nowak and Sigmund (1992). The formulation of this strategy is so simple that several authors including Takahashi (2007) have used a similar equilibrium in related literature. Since there is no synergy between the activities, the generous tit-for-tat Nash equilibrium must be belief-free, whereby each player is indifferent to an action choice at all times. This property is the driving force behind our characterization.

Since each player's generous tit-for-tat strategy does not depend on the signal for his own action, the Nash equilibrium property does not depend on whether the signals are public or private; approximate efficiency is achievable irrespective of whether monitoring is public or pricate. Significantly, the players' discount factors are not required to approach unity. This is in contrast with previous works by Ely and Välimäki (2002), Piccione (2002), and Matsushima (2004), who assumed that the players are as patient as possible.[3]

Matsushima (1991) showed that when monitoring is private and satisfies conditional independence, it is impossible for the collusion to be sustained by a pure Nash equilibrium strategy that is independent of irrelevant information. In contrast, this paper employs generous tit-for-tat strategies that are not pure, do not satisfy the independence of irrelevant information, and do not assume conditional independence.

By assuming symmetry, we show that whenever a generous tit-for-tat Nash equilibrium does not exist, no pure Nash equilibrium other than the repetition of the defective action choices exists quite generally; it might be impossible for the players to collude as long as they do not utilize more complicated mixed strategies than generous tit-for-tat strategies.

This paper proceeds as follows: Section 2 describes the model; Section 3 illustrates the characterization; and Section 4 discusses approximate efficiency.

2. The Model

We consider a long-run relationship between players 1 and 2 in the discrete and infinite time horizon. The component game is denoted by $(A_i, u_i)_{i\in\{1,2\}}$, where A_i denotes the set of all actions for each player $i \in \{1,2\}$, $a_i \in A_i$, $A \equiv A_1 \times A_2$, $a \equiv (a_1, a_2) \in A$, $u_i : A \to R$, and $u_i(a)$ implies the payoff for player i induced by $a \in A$. We assume that there is no synergy between their activities; each player $i's$ payoff has an additively separable form:

$$u_i(a) = v_i(a_i) + w_i(a_j) \quad \text{for all} \quad a \in A.$$

Two random signals $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ occur after their action choices are made, where Ω_i denotes the set of possible ω_i, $\omega = (\omega_1, \omega_2)$, and $\Omega = \Omega_1 \times \Omega_2$. A

[3] Sekiguchi (1997) is an exception. For general surveys on repeated games, see Mailath and Samuelson (2006), for instance.

signal profile $\omega \in \Omega$ is randomly determined according to a conditional probability function $f(\cdot | a) : \Omega \to R_+$. Let $f_i(\omega_i | a) \equiv \sum_{\omega_j \in \Omega_j} f(\omega | a)$, where $j \neq i$. On the basis of a no-synergy condition, we assume that $f_i(\omega_i | a)$ is independent of a_j; we denote $f_i(\omega_i | a_i)$ instead of $f_i(\omega_i | a)$ and use $\omega_i \in \Omega_i$ to denote *the signal for player* $i's$ *action.* At every period $t \in \{1, 2, ...\}$, each player i cannot directly observe the other player's action $a_j(t) \in A_j$. Instead, he observes $\omega_j(t) \in \Omega_j$ through which he imperfectly monitors it. Let $h(t) = (\omega(\tau))_{\tau=1}^{t}$ denote the signal history up to period t. Let $H = \{h(t) | t = 0,1,...\}$, where $h(0)$ implies null history.

We assume that each player i cannot observe $w_i(a_j)$; we regard $w_i(a_j)$ as the expected value of payoff induced by the signal for the opponent j's action $w_i^*(\omega_j)$, i.e., $w_i(a_j) = \sum_{\omega_j \in \Omega_j} w_i^*(\omega_j) f_j(\omega_j | a_j)$.

The formulation of repeated game with imperfect monitoring in the above manner assumes that monitoring is public in that each player can observe both signals ω_1 and ω_2. However, the argument of this paper will not depend on whether each player can observe the signal for his (or her) own action, i.e., whether monitoring is public or private.

Let $\alpha_i : A_i \to [0,1]$ denote a mixed action for player i. Let Δ_i denote the set of all mixed actions for player i. Let $\delta_i \in (0,1)$ denote the discount factor for player i, where we permit that the discount factors of both players could be different. Player $i's$ strategy is defined as $\sigma_i : H \to \Delta_i$, where he makes action choice $a_i(t)$ at each period t according to $\sigma_i(h(t-1)) \in \Delta_i$. Let Σ_i denote the set of all strategies for player i, $\sigma = (\sigma_1, \sigma_2)$, and $\Sigma = \Sigma_1 \times \Sigma_2$. The *payoff* for player i induced by $\sigma \in \Sigma$ is given by

$$U_i(\sigma) \equiv (1-\delta_i) E[\sum_{\tau=1}^{\infty} \delta_i^{\tau-1} u_i(a(\tau)) | \sigma],$$

where $E[\cdot | \sigma]$ denotes the expectation operator. A strategy profile $\sigma \in \Sigma$ is said to be a *Nash equilibrium* if

$$U_i(\sigma) \geq U_i(\sigma_i', \sigma_j) \text{ for all } i \in \{1,2\} \text{ and all } \sigma_i' \in \Sigma_i .$$

Let us specify the repeated prisoners' dilemma model: for each $i \in \{1,2\}$,

$$A_i = \{c_i, d_i\}, \quad \Omega_i = \{0,1\}, \quad f_i(1 | c_i) = p_i, \quad f_i(1 | d_i) = q_i,$$

$$v_i(c_i) = -X_i, \quad v_i(d_i) = 0, \quad w_i(c_j) = Y_i, \text{ and } w_i(d_j) = 0,$$

where $0 < q_i < p_i < 1$, and

(1) $$Y_i > X_i > 0 .$$

We call c_i and d_i the cooperative action and defective action respectively. It costs player i X_i for his cooperative action choice, but this choice gives the other player $j \neq i$ the benefit Y_j. It is important to note that we do not assume that Y_j is greater than X_i; the cooperative action choice does not necessarily improve welfare. However, inequality (1) guarantees that payoff vector $(Y_1 - X_1, Y_2 - X_2)$ induced by (c_1, c_2) is efficient and better than payoff vector $(0,0)$ that is induced by (d_1, d_2). We call 1 and 0 the good signal and bad signal respectively. Inequality $p_i > q_i$ implies that the probability of $\omega_i = 1$ is greater when player i makes the cooperative action choice than otherwise.

A strategy $\sigma_i \in \Sigma_i$ for each player $i \in \{1,2\}$ is said to be *generous tit-for-tat* if $s_i \in [0,1]$ such that for every $t \geq 1$ and every $h(t-1) \in H$

$$\sigma_i(h(t-1))(c_i) = 1 \text{ if either } \omega_j(t-1) = 1 \text{ or } t = 1,$$

and

$$\sigma_i(h(t-1))(c_i) = s_i \text{ if } \omega_j(t-1) = 0 .$$

We denote a generous tit-for-tat strategy by s_i for player i; he certainly selects c_i at period 1. At every period $t \geq 2$, he selects c_i if he observes $\omega_j(t-1) = 1$ at the previous period $t-1$. When he detects $\omega_j(t-1) = 0$ at the previous period, he selects d_i with probability $1 - s_i$; he selects c_i with probability s_i even after detecting the bad signal. Let $s = (s_1, s_2) \in [0,1]^2$.

3. Characterization

The following theorem shows a necessary and sufficient condition for the existence of generous tit-for-tat Nash equilibrium, which, if it exists, is unique.

Theorem 1: *A generous tit-for-tat strategy profile* s *is a Nash equilibrium if and only if for each* $i \in \{1, 2\}$,

(2) $$\delta_i \geq \frac{X_i}{(p_i - q_i)Y_i}, \text{ and}$$

(3) $$s_j = 1 - \frac{X_i}{\delta_i(p_i - q_i)Y_i}.$$

The payoff for each player i *induced by* s *is given by*

(4) $$U_i(s) = Y_i - \frac{1 - q_i}{p_i - q_i} X_i.$$

Proof: Selecting c_i costs player i X_i at the current period, whereas he gains Y_i with probability $p_i + (1 - p_i)s_j$ and nothing with probability $(1 - p_i)(1 - s_j)$ from the other player's response at the next period. This holds irrespective of period and history, because there is no synergy. He is incentivized to select both c_i and d_i at the same time; indifference between these action choices must be a necessary and sufficient condition:

$$-X_i + \delta_i Y_i\{p_i + (1 - p_i)s_j\} = \delta_i Y_i\{q_i + (1 - q_i)s_j\},$$

which is equivalent to $s_j = 1 - \frac{X_i}{\delta_i(p_i - q_i)Y_i}$ (equality (3)). Since $s_j \geq 1$, it follows that $\frac{X_i}{\delta_i(p_i - q_i)Y_i} \leq 1$ (inequality (2)) must hold. Since player i is indifferent to the action choice at all times, we can calculate his payoff by letting him select c_i at all times and the other player follow s_j:

$$U_i(s) = (1 - \delta_i)[Y_i + \frac{\delta_i Y_i\{p_i + (1 - p_i)s_j\}}{1 - \delta_i}] - X_i$$

$$= (1-\delta_i)[Y_i + \frac{\delta_i Y_i [p_i + (1-p_i)\{1 - \frac{X_i}{\delta_i (p_i - q_i) Y_i}\}]}{1-\delta_i}] - X_i$$

$$= Y_i - \frac{1-q_i}{p_i - q_i} X_i ,$$

which is equality (4)

Q.E.D.

Note that the players' payoffs are determined independent of their discount factors, while each player $i's$ generosity—the probability s_i that he selects c_i even after detecting the bad signal—depends on the other player's discount factor. The more patient one player is, the more generous is the other one. We do not require $Y_j > X_i$; a player prefers interlinking the other player's activity with his activity that may not contribute to the welfare alone but is easy to monitor.

The Nash equilibrium property does not depend on whether the signals are public or private. We can prove the same statement as Theorem 1, even if we restrict attentions to strategies for each player that depend only on the histories of his (or her) own actions and the signals for the other player's action, i.e., even if monitoring is assumed to be private. Each player i does not need to know whether the signal for his own action is good or bad.

4. Approximate Efficiency

The following theorem presents a sufficient condition under which the generous tit-for-tat Nash equilibrium approximately induces efficiency when monitoring is almost perfect.

Theorem 2: *Suppose that*

(5) $$\delta_i > \frac{X_i}{(1-q_i)Y_i} \quad \textit{for each} \quad i \in \{1,2\}$$

and p_i *is sufficiently close to unity for each* $i \in \{1,2\}$. *Then, a generous tit-for-tat*

Nash equilibrium that approximately induces $(Y_1 - X_1, Y_2 - X_2)$ *exists.*

Proof: Inequality (5), along with the supposition that p_i is close to unity, implies inequality (2). Theorem 1 implies that the strategy profile specified by (3) is a Nash equilibrium. Since p_i is close to unity, $Y_i - \frac{1-q_i}{p_i - q_i} X_i$ approximates $Y_i - X_i$.

Q.E.D.

Theorem 2 implies that the payoff vector induced by the generous tit-for-tat Nash equilibrium converges in efficiency as p_1 and p_2 approach unity. This convergence holds for any combination of their discount factors satisfying inequalities (5). Moreover, this convergence holds even if monitoring is private.

Let us assume that the model is symmetric:

$$X_1 = X_2,\ Y_1 = Y_2,\ p_1 = p_2,\ q_1 = q_2, \text{ and } \delta_1 = \delta_2 .$$

We omit the subscripts (e.g., $X = X_i$). The following proposition shows that inequality (5) is almost necessary for the existence of pure strategy Nash equilibrium that is not a repetition of the defective action choices, provided q_i is close to zero.

Proposition 3: *If the model is symmetric and*

(6) $$\delta < \frac{X}{Y},$$

then, the repetition of the defective action choices is the only pure strategy Nash equilibrium.

Proof: Consider an arbitrary pure strategy Nash equilibrium σ. Suppose that there exist $t \geq 1$ and $h(t-1) \in H$ such that

$$\sigma_i(h(t-1))(c_i) = 1 \text{ for each } i \in \{1,2\}.$$

Let B_i denote the continuation payoff at period $t+1$ associated with $h(t-1)$ induced by σ. Note that the future punishment for the selection of d_i instead of c_i is at most B_i minus the mini-max payoff (zero), i.e., is at most B_i. Hence, the incentive constraints imply that for each $i \in \{1,2\}$,

(7) $$(1-\delta)X \le \delta B_i .$$

Since $B_1 + B_2 \le 2(Y - X)$, it follows from (7) that $\delta \ge \frac{X}{Y}$. This contradicts (6).

Suppose that there exists $i \in \{1,2\}$, $t \ge 1$, and $h(t-1) \in H$ such that

$$\sigma_i(h(t-1))(c_i) = 1 .$$

Player $i's$ incentive constraint implies (7). From the above argument it follows that the players never select (c_1, c_2). Hence, the summation of their continuation payoffs is at most $Y - X$. Since the other player's payoff must be at least zero, it follows that $B_i \le Y - X$, which along with (7) implies $\delta \ge \frac{X}{Y}$. This contradicts (6).

Q.E.D.

If p_i and $1 - q_i$ are both close to unity, then inequality (6) mostly contradicts inequality (2). Hence, if monitoring is almost perfect, we can hardly expect the presence of collusive pure strategy Nash equilibrium when no generous tit-for-tat Nash equilibrium exists.

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