CIRJE-F-841

A Semi-group Expansion for Pricing Barrier Options

Takashi Kato
Osaka University

Akihiko Takahashi
University of Tokyo

Toshihiro Yamada
Mitsubishi UFJ Trust Investment Technology Institute Co.,Ltd. (MTEC)

February 2012;
Revised in March 2012, April 2012, June 2013, and August 2014

A Semi-group Expansion for Pricing Barrier Options *

Takashi Kato† Akihiko Takahashi‡ Toshihiro Yamada§¶

August 11, 2014

Abstract

This paper presents a new asymptotic expansion method for pricing continuously monitoring barrier options. In particular, we develops a semi-group expansion scheme for the Cauchy-Dirichlet problem in the second-order parabolic partial differential equations (PDEs) arising in barrier option pricing. As an application, we propose a concrete approximation formula under a stochastic volatility model and demonstrate its validity by some numerical experiments.

Keywords : Barrier options, Asymptotic expansion, Stochastic volatility model, Semi-group representation, Cauchy-Dirichlet problem.

1 Introduction

Since the Merton's seminal work ([15]) barrier options have been quite popular and important products in both academics and financial business for the last four decades. In particular, fast and accurate computation of their prices and Greeks is highly desirable in the risk management, which is a tough task under the finance models commonly used in practice. Thus, it has been one of the central issues in the mathematical finance community. Among various approaches to attacking the problem, this paper proposes a new semi-group expansion scheme under general diffusion setting.

Firstly, let us note that the value of a continuously monitoring down-and-out barrier option is expressed as the following form under the so called risk-neutral probability measure:

$$C_{\mathrm{Barrier}}(T,x) = \mathrm{E}\left[e^{-\int_0^T c(X_r^x)dr} f(X_T^x) 1_{\{\tau > T\}}\right] = \mathrm{E}[e^{-\int_0^T c(X_r^x)dr} f(X_T^x) 1_{\{\min_{t\in[0,T]} X_t > L\}}]. \quad (1.1)$$

* *Forthcoming in* **International Journal of Stochastic Analysis**. We are very grateful to Professor Seisho Sato in the University of Tokyo for his suggestions to our numerical computation. Also we would like to thank the anonymous referees for their valuable comments and suggestions.

† Division of Mathematical Science for Social Systems, Graduate School of Engineering Science, Osaka University, 1-3, Machikaneyama-cho, Toyonaka, Osaka 560-8531, Japan, E-mail: kato@sigmath.es.osaka-u.ac.jp

‡ Graduate School of Economics, The University of Tokyo, 7-3-1 Hongo, Bunkyo, Tokyo, 113-0033 Japan.

§ Mitsubishi UFJ Trust Investment Technology Institute Co.,Ltd. (MTEC), 2-6, Akasaka 4-Chome, Minato, Tokyo, 107-0052 Japan, E-mail: yamada@mtec-institute.co.jp

¶ Graduate School of Economics, The University of Tokyo, 7-3-1 Hongo, Bunkyo, Tokyo, 113-0033 Japan.

Mathematical Subject Classification (2010) 35B20, 35C20, 91G20

Here, $T > 0$ is a maturity of the option and f is an option payoff function at maturity. $(X_t^x)_t$ denotes a vector process with initial value x under the risk-neutral probability measure, which includes a price process of the underlying asset (usually given as the solution of a certain stochastic differential equation (SDE)). Also, c stands for the risk-free interest rate process. Moreover, L stands for a constant lower barrier, that is $L < x$, and τ is the hitting time to L:

$$\tau = \inf\{t \in [0, T] : X_t^x \leq L\}. \tag{1.2}$$

It is well-known that a possible approach in computation of $C_{\mathrm{Barrier}}(T, x)$ is the Euler-Maruyama scheme, which stores the sample paths of the process $(X_t^x)_t$ through an n-time discretization with the step size T/n. When applying this scheme to pricing a continuously monitoring barrier option, one kills the simulated process, say $(\bar{X}_{t_i}^x)_i$ if $\bar{X}_{t_i}^x$ exits from the domain (L, ∞) until the maturity T. The usual Euler-Maruyama scheme is *suboptimal* since it does not control the diffusion paths between two successive dates t_i and t_{i+1}: the diffusion paths could have crossed the barriers and come back to the domain without being detected. It is also known that the error between $C_{\mathrm{Barrier}}(T, x)$ and $\bar{C}_{\mathrm{Barrier}}(T, x)$, the barrier option price obtained by the Euler-Maruyama scheme is of order $\sqrt{T/n}$, as opposed to the order T/n for standard plain-vanilla options. (See [7]) Thus, various Monte-Carlo schemes have been proposed for improving the order of the error. (See [17] for instance.)

One of the other tractable approaches for calculating $C_{\mathrm{Barrier}}(T, x)$ is to derive an analytical approximation. If we obtain an accurate approximation formula, it is a powerful tool for pricing continuously monitoring barrier options because we need not rely on Monte-Carlo simulations anymore. However, from a mathematical viewpoint, deriving an approximation formula by applying stochastic analysis is not an easy task since the Malliavin calculus cannot be directly applied. It is due to the non-existence of the Malliavin derivative $D_t\tau$ (see [4]) and to the fact that the minimum (maximum) process of the Brownian motion has only the first-order differentiability in the Malliavin sense. Thus, neither approach in [12] nor in [20] can be applied directly to valuation of continuously monitoring barrier options, while they are applicable to pricing discrete barrier options. (See [19] for the detail.)

This paper proposes a new general method for the approximation of barrier option prices. Particularly, our objective is to pricing barrier options when the vector process of the underlying state variables is described by the following perturbed SDE under a filtered complete probability space with the risk-neutral probability measure, which will be concretely defined in the begging of the next section:

$$\begin{cases} dX_t^{\varepsilon,x} = b(X_t^{\varepsilon,x}, \varepsilon)dt + \sigma(X_t^{\varepsilon,x}, \varepsilon)dB_t, \\ X_0^{\varepsilon,x} = x, \end{cases} \tag{1.3}$$

where ε is a small parameter. In this case, the barrier option price (1.1) is characterized as a solution of the Cauchy-Dirichlet problem:

$$\begin{cases} \dfrac{\partial}{\partial t}u^\varepsilon(t, x) + \mathscr{L}^\varepsilon u^\varepsilon(t, x) = 0, & (t, x) \in [0, T) \times (L, \infty), \\ u^\varepsilon(T, x) = f(x), & x > L, \\ u^\varepsilon(t, L) = 0, & t \in [0, T], \end{cases} \tag{1.4}$$

where the differential operator $\mathscr{L}^\varepsilon$ is determined by the diffusion coefficients b and σ with the risk-free interest rate c, which will be explicitly defined in the next section.

Next, we introduce an asymptotic expansion formula:

$$u^\varepsilon(t,x) = u^0(t,x) + \varepsilon v_1^0(t,x) + \cdots + \varepsilon^{n-1} v_{n-1}^0(t,x) + O(\varepsilon^n), \tag{1.5}$$

where O denotes the Landau symbol. The function $u^0(t,x)$ is the solution of (1.4) with $\varepsilon = 0$: if $b(x,0)$ and $\sigma(x,0)$ have some simple forms such as constants (as in the Black-Scholes model), we already know the closed form of $u^0(t,x)$ and hence obtain the price. Then, we are able to get the approximate value for $u^\varepsilon(t,x)$ through evaluation of the coefficient functions $v_1^0(t,x), \ldots, v_{n-1}^0(t,x)$. In fact, they are also characterized as the solution of a certain PDE with the Dirichlet condition. By formal asymptotic expansions, (1.5) above and (1.6) below,

$$\mathscr{L}^\varepsilon = \mathscr{L}^0 + \varepsilon \tilde{\mathscr{L}}_1^0 + \cdots + \varepsilon^{n-1} \tilde{\mathscr{L}}_{n-1}^0 + \cdots , \tag{1.6}$$

we can derive the following PDE which $v_k^0(t,x)$ satisfies:

$$\begin{cases} \dfrac{\partial}{\partial t} v_k^0(t,x) + \mathscr{L}^0 v_k^0(t,x) + g_k^0(t,x) = 0, & (t,x) \in [0,T) \times (L,\infty), \\ v_k^0(T,x) = 0, & x > L, \\ v_k^0(t,L) = 0, & t \in [0,T], \end{cases} \tag{1.7}$$

where $g_k^0(t,x)$ will be given explicitly in Section 2. Moreover, by applying the Feynman-Kac approach to the PDE (1.7), we obtain a semi-group representation of v_k^0. That is, for each $k = 1, \ldots, n-1$,

$$\begin{aligned} & v_k^0(T-t,x) \\ &= \sum_{l=1}^{k} \sum_{(\beta^i)_{i=1}^l \subset \mathbb{N}^l, \sum_i \beta^i = k} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} P_{t-t_1}^D \tilde{\mathscr{L}}_{\beta^1}^0 P_{t_1-t_2}^D \tilde{\mathscr{L}}_{\beta^2}^0 \cdots P_{t_{l-1}-t_l}^D \tilde{\mathscr{L}}_{\beta^l}^0 P_{t_l}^D f(x) dt_l \cdots dt_1, \end{aligned} \tag{1.8}$$

where $(P_t^D)_t$ is a semi-group defined in Section 2. We will justify the above argument in a mathematically rigorous manner in Section 2

The theory of the Cauchy-Dirichlet problem for this kind of the second order parabolic PDE is well understood for the case of bounded domains (see [5], [6] and [14] for instance). As for an unbounded domain case such as (1.4), [18] provides the existence and uniqueness results for a solution of the PDE and the Feynman-Kac type formula, the part of which will be cited as Theorem 1 in Section 2. However, some mathematical difficulty exists for applying the results of [18] to the PDE (1.7). More precisely, the function $g_k^0(t,x)$ may be divergent at $t = T$. Hence, in order to obtain an asymptotic expansion (1.5), we generalize the result of [18] and the argument of the Feynman–Kac representation. Furthermore, we derive a new representation (1.8) for $v_k^0(t,x)$ by using the semi-group $(P_t^D)_t$. We notice that such a form is convenient for evaluation of $v_k^0(t,x)$ in concrete examples.

We also apply our method to pricing a barrier option in a stochastic volatility model. Then, as an example of (1.8) we obtain a new approximation formula of the barrier option price $C_{\text{Barrier}}^{SV,\varepsilon}$ under a stochastic volatility model as follows: for the initial value of the logarithmic underlying price x, the maturity T and the lower barrier L,

$$C_{\text{Barrier}}^{SV,\varepsilon}(T,e^x) \;=\; \mathrm{E}\left[e^{-cT} f(S_T^{\varepsilon,x}) 1_{\{\min_{0\le t\le T} S_t^\varepsilon > L\}}\right]$$

$$\simeq \ P_T^D \bar{f}(x) + \varepsilon \int_0^T P_{T-r}^D \tilde{\mathscr{L}}_1^0 P_r^D \bar{f}(x) dr,$$

where $(S_t^{\varepsilon,x})_t$ is the underlying asset price process, f is a payoff function and $\bar{f}(x) = f(e^x)$ and the expectation is taken under the risk-neutral probability measure. Here, $P_T^D \bar{f}(x)$ is regarded as the down-and-out barrier option price in the Black-Scholes model. Moreover, we confirm practical validity of our method through a numerical example given in Section 3.

Finally, we remark that there exist the previous works on barrier option pricing such as [2], [3], [8], [9], which start with some specific models (e.g. Black-Scholes model or some type of fast mean-reversion model), and derive approximation formulas for discretely or continuously monitoring barrier option prices. Our approach is to firstly develop a general semi-group expansion scheme for the Cauchy-Dirichlet problem under multi-dimensional diffusion setting; then as an application, we provide a new approximation formula under a certain class of stochastic volatility model.

The organization of this paper is as follows: the next section firstly prepares the existence and uniqueness result for the Cauchy-Dirichlet problem in the second-order parabolic PDE associated with barrier option pricing. Then, we present our main result for an asymptotic expansion of barrier option prices. Section 3 shows numerical examples under a stochastic volatility model. Section 4 concludes. Finally, Appendix provides the proofs of the results in the main text.

2 Asymptotic Expansion of Barrier Option Price

2.1 Main Results

This subsection states the setup and our main results. The relevant assumptions $[A]$–$[H]$ will be explained in the next subsection.

Suppose first that a filtered complete probability space $(\Omega, \mathcal{F}, Q, \{\mathcal{F}_t\}_{t\in[0,T]})$ is given, where Q denotes the risk-neutral probability measure, the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions and $T > 0$ is some fixed time horizon. Then, the d-dimensional underlying state variable is described by the following perturbed SDE:

$$\begin{cases} dX_t^{\varepsilon,x} = b(X_t^{\varepsilon,x}, \varepsilon)dt + \sigma(X_t^{\varepsilon,x}, \varepsilon)dB_t, \\ X_0^{\varepsilon,x} = x, \end{cases} \tag{2.1}$$

where B is an m-dimensional Brownian motion and ε is a small parameter. Let $b : \mathbb{R}^d \times I \longrightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times I \longrightarrow \mathbb{R}^d \otimes \mathbb{R}^m$ be Borel measurable functions ($d, m \in \mathbb{N}$) where I is an interval on $\mathbb{R}$ including the origin 0, for instance $I = (-1, 1)$.

We consider the SDE (2.1) for any $x \in \mathbb{R}^d$ and $\varepsilon \in I$. We note that Assumption [A] introduced in the next subsection guarantees the existence and uniqueness of a solution of (2.1). We also remark that at least one element of $X^{\varepsilon,x}$ stands for the underlying asset price process of a barrier option.

Next, we are interested in evaluation of the following barrier option price: for a small ε,

$$\begin{aligned} & u^\varepsilon(t,x) \\ = \ & \mathrm{E}\left[\exp\left(-\int_0^{T-t} c(X_r^{\varepsilon,x}, \varepsilon)dr\right) f(X_{T-t}^{\varepsilon,x}) 1_{\{\tau_D(X^{\varepsilon,x}) \geq T-t\}}\right], \quad (t,x) \in [0,T] \times \bar{D} \end{aligned} \tag{2.2}$$

for Borel measurable functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, $c : \mathbb{R}^d \times I \longrightarrow \mathbb{R}$ and a domain $D \subset \mathbb{R}^d$.

Also, $\bar{D} \subset \mathbb{R}^d$ is the closure of D and $\tau_D(w)$, $w \in C([0,T];\mathbb{R}^d)$ stands for the first exit time from D, that is

$$\tau_D(w) = \inf\{t \in [0,T]; w(t) \notin D\}.$$

We also remark that the expectation operator $\mathrm{E}[\cdot]$ is taken under the risk-neutral probability measure Q, and that $e^{-\int_0^{T-t} c(X_r^{\varepsilon,x},\varepsilon)dr}$ represents the discount factor with the risk-free interest rate process c.

Let us define a second order differential operator $\mathscr{L}^\varepsilon$ by

$$\mathscr{L}^\varepsilon = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x,\varepsilon)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(x,\varepsilon)\frac{\partial}{\partial x^i} - c(x,\varepsilon),$$

where $a^{ij} = \sum_{k=1}^{d} \sigma^{ik}\sigma^{jk}$. We consider the following Cauchy-Dirichlet problem for a PDE of parabolic type:

$$\begin{cases} \dfrac{\partial}{\partial t}u^\varepsilon(t,x) + \mathscr{L}^\varepsilon u^\varepsilon(t,x) = 0, & (t,x) \in [0,T) \times D, \\ u^\varepsilon(T,x) = f(x), & x \in D, \\ u^\varepsilon(t,x) = 0, & (t,x) \in [0,T] \times \partial D. \end{cases} \tag{2.3}$$

Under the assumptions [A]-[E] stated in the next subsection, we have the following existence and uniqueness result due to Theorem 3.1 in [18].

Theorem 1. *Assume* [A]–[E] *which are given in Section 2.2. For each* $\varepsilon \in I$, $u^\varepsilon(t,x)$ *defined with formula (2.2) is a (classical) solution of* (2.3) *and*

$$\sup_{(t,x)\in[0,T]\times\bar{D}} |u^\varepsilon(t,x)|/(1+|x|^{2m}) < \infty. \tag{2.4}$$

Moreover, if $w^\varepsilon(t,x)$ *is also a solution of* (2.3) *satisfying the growth condition*

$$\sup_{(t,x)\in[0,T]\times\bar{D}} |w^\varepsilon(t,x)|/(1+|x|^{2m'}) < \infty$$

for some $m' \in \mathbb{N}$, *then* $u^\varepsilon = w^\varepsilon$.

Our main purpose is to present an asymptotic expansion of the barrier option price $u^\varepsilon(t,x)$:

$$u^\varepsilon(t,x) = u^0(t,x) + \varepsilon v_1^0(t,x) + \cdots + \varepsilon^{n-1}v_{n-1}^0(t,x) + O(\varepsilon^n), \quad \varepsilon \to 0. \tag{2.5}$$

Here, the coefficient functions $v_k^0(t,x)$, $k = 1,\ldots,n-1$ are (formally) given as the solution of

$$\begin{cases} \dfrac{\partial}{\partial t}v_k^0(t,x) + \mathscr{L}^0 v_k^0(t,x) + g_k^0(t,x) = 0, & (t,x) \in [0,T) \times D, \\ v_k^0(T,x) = 0, & x \in D, \\ v_k^0(t,x) = 0, & (t,x) \in [0,T] \times \partial D, \end{cases} \tag{2.6}$$

where $g_k^0(t,x)$ is given inductively by

$$g_k^0(t,x) = \tilde{\mathscr{L}}_k^0 u^0(t,x) + \sum_{l=1}^{k-1} \tilde{\mathscr{L}}_{k-l}^0 v_l^0(t,x). \tag{2.7}$$

Here, $\tilde{\mathscr{L}}_k^0$ is defined as follows:

$$\tilde{\mathscr{L}}_k^0 = \frac{1}{k!}\left\{ \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^k a^{ij}}{\partial \varepsilon^k}(x,0)\frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d \frac{\partial^k b^i}{\partial \varepsilon^k}(x,0)\frac{\partial}{\partial x^i} - \frac{\partial^k c}{\partial \varepsilon^k}(x,0)\right\}. \tag{2.8}$$

Then, we can show the next result, whose proof is given in Section 6.1 of Appendix.

Theorem 2. *Assume [A]–[H] which are stated in Section 2.2.. Then, for each $k = 1, \dots, n-1$, v_k^0 is the classical solution of (2.6) and satisfies*

$$|v_k^0(t,x)| \le C_k(1 + |x|^{2m_k}), \quad (t,x) \in [0,T] \times \mathbb{R}^d \tag{2.9}$$

for some $C_k, m_k > 0$.

Note that the uniqueness of the solutions of (2.6) follows from the same arguments as in the proof of Theorem 5.7.6 in [10]. That is, we obtain the next proposition.

Proposition 1. *For any function g which has a polynomial growth rate in x uniformly in t, a classical solution of (2.6) is unique in the following sense: if v and w are classical solutions of (2.6) and $|v(t,x)| + |w(t,x)| \le C(1 + |x|^{2m})$ for some $C, m > 0$, then $v = w$.*

Now, we are able to state our first main result on the asymptotic expansion. The proof is given in Section 6.2 of Appendix.

Theorem 3. *Assume [A]–[H] which are given in Section 2.2. There are positive constants C_n and $\tilde{m}_n$ which are independent of ε such that*

$$\left| u^\varepsilon(t,x) - \left(u^0(t,x) + \sum_{k=1}^{n-1} \varepsilon^k v_k^0(t,x)\right)\right| \le C_n(1 + |x|^{2\tilde{m}_n})\varepsilon^n, \quad (t,x) \in [0,T] \times \bar{D}.$$

Next, we construct a semi-group corresponding to $(X_t^{0,x})_t$ (that is, $(X_t^{\varepsilon,x})_t$ with $\varepsilon = 0$) and D. Then, based on this semi-group we can obtain more explicit representation for the coefficient function $v_k^0(t,x)$ than the right hand side of (2.15) in Assumption $[H]$, which will appear in the following subsection.

Let $C_b^0(\bar{D})$ be the set of bounded continuous functions $f : \bar{D} \longrightarrow \mathbb{R}$ such that $f(x) = 0$ on ∂D. Obviously, $C_b^0(\bar{D})$ equipped with the sup-norm becomes a Banach space.

For $t \in [0,T]$ and $f \in C_b^0(\bar{D})$, we define $P_t^D f : \bar{D} \longrightarrow \mathbb{R}$ by

$$P_t^D f(x) = \mathrm{E}\left[\exp\left(-\int_0^t c(X_v^{0,x},0)dv\right) f(X_t^{0,x}) 1_{\{\tau_D(X^{0,x}) \ge t\}}\right], \tag{2.10}$$

where $c(x,0)$ is non-negative. We notice that $P_t^D f(x)$ is equal to $u^0(T-t,x)$ with the payoff function f. Then, we have the following result:

Proposition 2. *Under the assumptions [A]–[E] stated in Section 2.2, the mapping* $P_t^D : C_b^0(\bar{D}) \longrightarrow C_b^0(\bar{D})$ *is well-defined and* $(P_t^D)_{0\le t\le T}$ *is a contraction semi-group.*

Proof. Let $f \in C_b^0(\bar{D})$. The relations $P_0^D f = f$, $P_t^D f|_{\partial D} = 0$ and $\sup_{\bar{D}} |P_t^D f| \le \sup_{\bar{D}} |f|$ are obvious. The continuity of $P_t^D f$ is by Lemma 4.3 in [18]. The semi-group property is verified by a straightforward calculation. ■

Remark 1. *Note that* $(P_t^D)_t$ *also has the semi-group property on the set* $C_p^0(\bar{D})$ *of continuous functions* f*, each of which has a polynomial growth rate and satisfies* $f(x) = 0$ *on* ∂D*.*

Finally, we show our second main result on the semi-group representation of the coefficient function v_k^0 in the expansion, whose proof is given in Section 6.3 in Appendix.

Theorem 4. *Under Assumptions [A]–[H] given in Section 2.2, for each* $k = 1, \ldots, n-1$

$$\begin{aligned} & v_k^0(T-t,x) \\ &= \sum_{l=1}^{k} \sum_{(\beta^i)_{i=1}^l \subset \mathbb{N}^l, \sum_i \beta^i = k} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} P_{t-t_1}^D \tilde{\mathscr{L}}_{\beta^1}^0 P_{t_1-t_2}^D \tilde{\mathscr{L}}_{\beta^2}^0 \cdots P_{t_{l-1}-t_l}^D \tilde{\mathscr{L}}_{\beta^l}^0 P_{t_l}^D f(x) dt_l \cdots dt_1. \end{aligned} \tag{2.11}$$

2.2 Assumptions

This subsection introduces a series of the assumptions necessary for our main results stated in the previous subsection. Particularly, the assumptions [A]–[E] are relevant for Theorem 1, that is the existence and uniqueness result of the PDE (2.3), while [F]–[H] are additional assumptions necessary for the asymptotic expansion results, that is Theorem 2-Theorem 4.

[A] There is a positive constant A_1 such that

$$|\sigma^{ij}(x,\varepsilon)|^2 + |b^i(x,\varepsilon)|^2 \le A_1(1+|x|^2), \quad x \in \mathbb{R}^d, \ \varepsilon \in I, \ i,j = 1,\ldots,d.$$

Moreover, for each $\varepsilon \in I$ it holds that $\sigma^{ij}(\cdot,\varepsilon), b^i(\cdot,\varepsilon) \in \mathcal{L}$ for $i,j = 1,\ldots,d$, where $\mathcal{L}$ is the set of locally Lipschitz continuous functions defined on $\mathbb{R}^d$:

$$\begin{aligned} \mathcal{L} &= \{f \in C(\mathbb{R}^d;\mathbb{R}); \text{ for any compact set } K \subset \mathbb{R}^d, \\ &\qquad \exists C_K > 0 \text{ such that } |f(x)-f(y)| \le C_K|x-y|, x,y \in K\} \end{aligned}$$

Remark 2. *Note that under [A], the existence and uniqueness of a solution of (2.1) are guaranteed on any filtered probability space equipped with a standard* m*-dimensional Brownian motion, and Corollary 2.5.12 in [11] and Lemma 3.2.6 in [16] imply*

$$\mathrm{E}[\sup_{0\le r\le t} |X_r^{\varepsilon,x} - x|^{2l}] \le C_l t^l e^{C_l t}(1+|x|^{2l}), \quad (t,x) \in [0,T]\times\mathbb{R}^d, \ l \in \mathbb{N} \tag{2.12}$$

for some $C_l > 0$*, which depends only on* l*,* A_1 *and* C_K*. Moreover,* $(X_r^x)_r$ *has the strong Markov property.*

[B] The function $f(x)$ is continuous on $\bar{D}$ and there are $C_f > 0$ and $m \in \mathbb{N}$ such that $|f(x)| \leq C_f(1 + |x|^{2m})$, $x \in \mathbb{R}^d$. Moreover, $f(x) = 0$ on $\mathbb{R}^d \setminus D$.

Remark 3. *The assumption [B] which corresponds to* **H2** *- (2) in [18] guarantees the continuity of a solution of (2.3) (if it exists) on the so called parabolic boundary* $\Sigma = \partial D \times [0, T) \cup \bar{D} \times \{T\}$. *For the details, see p.8 in [18].*

[C] $c(x, \varepsilon)$ is non-negative (i.e. $c(x, \varepsilon) \geq 0$). Moreover, for each $\varepsilon \in I$, it holds that $c(\cdot, \varepsilon) \in \mathcal{L}$.

[D] The boundary ∂D has the outside strong sphere property, that is, for each $x \in \partial D$ there is a closed ball E such that $E \cap D = \emptyset$ and $E \cap \bar{D} = \{x\}$.

Remark 4. *The assumption [D] provides the regularity of each point in* ∂D. *(c.f.[5]) Also, [18] points out that [D] with the ellipticity of the matrix* $(a^{ij}(x, \varepsilon))_{ij}$ *in [E] below gives*

$$P(\tau_D(X^{\epsilon,x}) = \tau_{\bar{D}}(X^{\epsilon,x})) = 1.$$

[E] The matrix $(a^{ij}(x, \varepsilon))_{ij}$ is locally elliptic in the sense that for each $\varepsilon \in I$ and compact set $K \subset \mathbb{R}^d$ there is a positive number $\mu_{\varepsilon,K}$ such that $\sum_{i,j=1}^{d} a^{ij}(x, \varepsilon)\xi^i\xi^j \geq \mu_{\varepsilon,K}|\xi|^2$ for any $x \in K$ and $\xi \in \mathbb{R}^d$.

Remark 5. *Note that although the condition [E] (local ellipticity) is necessary for the existence of classical solution of our PDE (See Remark 2.2 in [18]), the assumption can be removed through consideration of viscosity solutions rather than classical solutions by applying Theorem 8.2 in [1] and Theorems 4.4.3 and 7.7.2 in [16]. Note that we need the additional assumption such that* $I \subset [0, \infty)$ *by technical reason in this case.*

To study the asymptotic expansion, we put the following assumptions in addition to [A]–[E]. Firstly, by the next condition we can properly define $\mathscr{L}_k^0$, $k \in \mathbb{N}$ in (2.8) above.

[F] Let $n \in \mathbb{N}$. The functions $a^{ij}(x, \varepsilon)$, $b^i(x, \varepsilon)$ and $c(x, \varepsilon)$ are n-times continuously differentiable in ε. Furthermore, each of derivatives $\partial^k a^{ij}/\partial\varepsilon^k$, $\partial^k b^i/\partial\varepsilon^k$, $\partial^k c/\partial\varepsilon^k$, $k = 1, \ldots, n-1$, has a polynomial growth rate in $x \in \mathbb{R}^d$ uniformly in $\varepsilon \in I$.

To state the existence of the functions $v_k^0(t, x)$ in the asymptotic expansion (2.5), we first prepare the following set.

Definition 1. *The set* $\mathcal{H}^{m,p}$ *of* $g \in C([0, T) \times \bar{D})$ *is defined to satisfy the following condition: There is some* $M^g \in C([0, T)) \cap L^p([0, T), dt)$ *such that*

$$|g(t, x)| \leq M^g(t)(1 + |x|^{2m}), \quad t \in [0, T), \ x \in \bar{D}. \tag{2.13}$$

Given this definition of the set $\mathcal{H}^{m,p}$, we put the next condition on u^0.

[G] $u^0 \in \mathcal{G}^m$, where

$$\mathcal{G}^m = \Big\{u \in C^{1,2}([0, T) \times D) \cap C([0, T] \times \bar{D}) \ ; \quad \frac{\partial u}{\partial x^i} \in \mathcal{H}^{m,2}, \ \frac{\partial^2 u}{\partial x^i \partial x^j} \in \mathcal{H}^{m,1}, \ i, j = 1, \ldots, d\Big\}.$$

Now we examine the conditions necessary for the classical solution to the PDE (2.6).

Firstly, Let us start with the case of $k=1$. By the assumption [G], we have $g_1^0 \in \mathcal{H}^{m,1}$ for some $m \in \mathbb{N}$ by the definition of g_k^0 with $k=1$ in (2.7). Thus we can define

$$v_1^0(t,x) = \mathrm{E}\left[\int_0^{(T-t)\wedge\tau_D(X^{0,x})} \exp\left(-\int_0^r c(X_v^{0,x},0)dv\right) g_1^0(t+r,X_r^{0,x})dr\right]. \tag{2.14}$$

Therefore, if we assume that $v_1^0 \in C^{1,2}([0,T)\times D)$ we can show that v_1^0 is the solution of (2.6) with $k=1$, that is, we can confirm that

$$\frac{\partial}{\partial t}v_1^0(t,x) + \mathscr{L}^0 v_1^0(t,x) + g_1^0(t,x) = 0.$$

Note that the relations $v_1^0(T,\cdot)=0$ and $v_1^0=0$ on $[0,T]\times\partial D$ are obvious.

Next, let us give some comments on the smoothness of v_1^0. In many cases as in the Black–Scholes model (see (3.10) in Section 3) we can rewrite (2.14) as

$$v_1^0(t,x) = \int_0^{T-t}\int_D g_1^0(t+r,y)p(r,x,y)dydr$$

for some $p(r,x,y)$. Thus, if p has a "good" smoothness property, the smoothness of v_1^0 also holds such as

$$\frac{\partial}{\partial t}v_1^0(t,x) = -\lim_{s\to T}\int_D g_1^0(s,y)p(s-t,x,y)dy + \int_0^t\int_D \frac{\partial}{\partial t}g_1^0(t+r,y)p(r,x,y)dr,$$

if the limit in the right hand side exists, and

$$\begin{aligned}\frac{\partial}{\partial x^i}v_1^0(t,x) &= \int_0^t\int_D g_1^0(t+r,y)\frac{\partial}{\partial x^i}p(r,x,y)dr,\\ \frac{\partial^2}{\partial x^i\partial x^j}v_1^0(t,x) &= \int_0^t\int_D g_1^0(t+r,y)\frac{\partial^2}{\partial x^i\partial x^j}p(r,x,y)dr.\end{aligned}$$

Moreover, if v_1^0 is in $\mathcal{G}^{m_1}$ for some $m_1\in\mathbb{N}$, we also have $g_2^0 \in \mathcal{H}^{\tilde{m}_1,1}$ for some $\tilde{m}_1\in\mathbb{N}$ by the definition of g_k^0 with $k=2$ in (2.7). Then, we can define v_2^0 similarly as v_1^0. Furthermore, under some suitable smoothness conditions for v_2^0, which may be given by the smoothness property of $p(r,x,y)$, we are able to show that v_2^0 is the classical solution of (2.6) with $k=2$.

Thus, the observation above leads us to our final assumption.

[H] It holds that $v_k^0 \in \mathcal{G}^{m_n}$, $k=1,\dots,n-1$ for some $m_n\in\mathbb{N}$, where

$$v_k^0(t,x) = \mathrm{E}\left[\int_0^{(T-t)\wedge\tau_D(X^{0,x})} \exp\left(-\int_0^r c(X_v^{0,x},0)dv\right) g_k^0(t+r,X_r^{0,x})dr\right]. \tag{2.15}$$

3 Application to Barrier Option Pricing in Stochastic Volatility Environment

This section demonstrates the effectiveness of our method in stochastic volatility environment: Section 3.1 derives concrete approximation formulas, and Section 3.2 shows numerical examples.

3.1 Approximation of Barrier Option Prices in a Stochastic Volatility Model

We consider the following stochastic volatility model under the risk-neutral probability measure:

$$
\begin{aligned}
dS_t^\varepsilon &= (c-q)S_t^\varepsilon dt + \sigma_t^\varepsilon S_t^\varepsilon dB_t^1,\ S_0^\varepsilon = S, \\
d\sigma_t^\varepsilon &= \varepsilon\lambda(\theta-\sigma_t^\varepsilon)dt + \varepsilon\nu\sigma_t^\varepsilon(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2),\ \sigma_0^\varepsilon = \sigma,
\end{aligned}
\tag{3.1}
$$

where $c, q > 0$, $\varepsilon \in [0,1)$, $\lambda, \theta, \nu > 0$, $\rho \in [-1,1]$ and $B = (B^1, B^2)$ is a two dimensional Brownian motion. Here c and q represent a domestic interest rate and a foreign interest rate, respectively when we consider the currency options. Clearly, applying Itô's formula, we have its logarithmic process:

$$
\begin{aligned}
dX_t^\varepsilon &= (c-q-\frac{1}{2}(\sigma_t^\varepsilon)^2)dt + \sigma_t^\varepsilon dB_t^1,\ X_0^\varepsilon = x = \log S, \\
d\sigma_t^\varepsilon &= \varepsilon\lambda(\theta-\sigma_t^\varepsilon)dt + \varepsilon\nu\sigma_t^\varepsilon(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2),\ \sigma_0^\varepsilon = \sigma.
\end{aligned}
\tag{3.2}
$$

Also, its generator is expressed as

$$
\mathscr{L}^\varepsilon = \left(c-q-\frac{1}{2}\sigma^2\right)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} + \varepsilon\rho\nu\sigma^2\frac{\partial^2}{\partial x\partial\sigma} + \varepsilon\lambda(\theta-\sigma)\frac{\partial}{\partial\sigma} + \varepsilon^2\frac{1}{2}\nu^2\sigma^2\frac{\partial^2}{\partial\sigma^2} - c.
\tag{3.3}
$$

In this case, $\tilde{\mathscr{L}}_1^0$ which is defined by (2.8) with $k=1$ is given as

$$
\tilde{\mathscr{L}}_1^0 = \rho\nu\sigma^2\frac{\partial^2}{\partial x\partial\sigma} + \lambda(\theta-\sigma)\frac{\partial}{\partial\sigma}.
\tag{3.4}
$$

We will apply the asymptotic expansion in the previous section to (3.2) and give an approximation formula for a barrier option price, which is given under a risk-neutral probability measure as

$$
C_{\text{Barrier}}^{SV,\varepsilon}(T-t, e^x) = \mathrm{E}\left[e^{-c(T-t)}f(S_{T-t}^{\varepsilon,e^x})1_{\{\tau_{(L,\infty)}(S^{\varepsilon,e^x})>T-t\}}\right],
$$

where f stands for a payoff function, $L(< S)$ is a barrier price and the expectation is taken with respect to the risk-neutral probability measure.

Then, $u^\varepsilon(t,x) = C_{\text{Barrier}}^{SV,\varepsilon}(T-t, e^x)$ satisfies the following PDE:

$$
\begin{cases}
\left(\frac{\partial}{\partial t} + \mathscr{L}^\varepsilon\right)u^\varepsilon(t,x) = 0, & (t,x) \in (0,T] \times D, \\
u^\varepsilon(T,x) = \bar{f}(x), & x \in \bar{D}, \\
u^\varepsilon(t,l) = 0, & t \in [0,T].
\end{cases}
\tag{3.5}
$$

where $\bar{f}(x) = \max\{e^x - K, 0\}$, $D = (l, \infty)$ and $l = \log L$. We obtain the 0-th order u^0 as

$$
u^0(t,x) = P_{T-t}^D\bar{f}(x) = \mathrm{E}[e^{-c(T-t)}\bar{f}(X_{T-t}^{x,0})1_{\{\tau_D(X^{0,x})>T-t\}}].
\tag{3.6}
$$

Remark 6. *u^0 satisfies the PDE (3.5) with $\varepsilon = 0$. Although the condition [E] in Section 2 does not seem to be satisfied in this case, the volatility process $(\sigma_t^\varepsilon)_t$ becomes a constant $\sigma > 0$, and so (3.2) is reduced to a one-dimensional SDE. Then, (3.5) with $\varepsilon = 0$ becomes a non-degenerating PDE with fixed σ. Therefore, we need not take care of the lack of the condition [E] in this example.*

Setting $\alpha = c - q$, we note that $P_{T-t}^D \bar{f}(x) = C_{\text{Barrier}}^{BS}(T-t, e^x, \alpha, \sigma, L)$ is the price of the down-and-out barrier call option under the Black-Scholes model:

$$C_{\text{Barrier}}^{BS}(T-t, e^x, \alpha, \sigma, L) = C^{BS}\left(T-t, e^x, \alpha, \sigma\right) - \left(\frac{e^x}{L}\right)^{1-\frac{2\alpha}{\sigma^2}} C^{BS}\left(T-t, \frac{L^2}{e^x}, \alpha, \sigma\right). \quad (3.7)$$

Here, we recall that the price of the plain vanilla option under the Black-Scholes model is given as

$$C^{BS}\left(T-t, e^x, \alpha, \sigma\right) = e^{-q(T-t)} e^x N(d_1(T-t, x, \alpha)) - e^{-c(T-t)} K N(d_2(T-t, x, \alpha)), \quad (3.8)$$

where

$$\begin{aligned} d_1(t, x, \alpha) &= \frac{x - \log K + \alpha t}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}, \\ d_2(t, x, \alpha) &= d_1(t, x, \alpha) - \sigma\sqrt{t} \\ N(x) &= \int_{-\infty}^{x} n(y) dy, \\ n(y) &= \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}. \end{aligned}$$

Note also that

$$P(\tau_D(X^{0,x}) \geq t | X_t^{0,x}) = 1 - \exp\left(-\frac{2(x-l)(X_t^{0,x} - l)}{\sigma^2 t}\right) \quad \text{on } \{X_t^{0,x} > l\}.$$

Therefore, for $g \in C_p^0(\bar{D})$ we have

$$P_t^D g(x) = \mathrm{E}[P(\tau_D(X^{0,x}) \geq t | X_t^x) e^{-ct} g(X_t^{0,x}) 1_{\{X_t^{0,x} > l\}}] = \int_l^\infty e^{-ct} g(y) p(t, x, y) dy, \quad (3.9)$$

where

$$\begin{aligned} p(t, x, y) &= \frac{1}{\sqrt{2\pi\sigma^2 t}} (1 - e^{-\frac{2(x-l)(y-l)}{\sigma^2 t}}) e^{-\frac{(y-x-\mu t)^2}{2\sigma^2 t}}, \quad (3.10) \\ \mu &= \alpha - \sigma^2/2 = (c - q - \sigma^2/2). \end{aligned}$$

Then, we show the following main result in this section.

Theorem 5. *We obtain an approximation formula for the down-and-out barrier call option under the stochastic volatility model (3.1):*

$$C_{\text{Barrier}}^{SV,\varepsilon}(T, e^x) = C_{\text{Barrier}}^{BS}(T, e^x, \alpha, \sigma, L) + \varepsilon v_1^0(0, x) + O(\varepsilon^2), \quad (3.11)$$

where

$$v_1^0(0,x) = e^{-cT}\int_0^T\int_l^\infty \frac{1}{\sqrt{2\pi\sigma^2 s}}(1-e^{-\frac{2(x-l)(y-l)}{\sigma^2 s}})e^{-\frac{(y-x-(\alpha-\frac{1}{2}\sigma^2)s)^2}{2\sigma^2 s}}\vartheta(s,y)dyds, \tag{3.12}$$

$$\begin{aligned}
& \vartheta(t,x) \\
= \; & e^{\alpha(T-t)}\rho\nu\sigma e^x n(d_1(T-t,x,\alpha))(-d_2(T-t,x,\alpha)) \\
+ \; & 2e^{\alpha(T-t)}\rho\nu\alpha\left(\frac{e^x}{L}\right)^{-\frac{2\alpha}{\sigma^2}} Ln(c_1(T-t,x,\alpha))\sqrt{T-t} \\
- \; & e^{\alpha(T-t)}\rho\nu\sigma\left(\frac{e^x}{L}\right)^{-\frac{2\alpha}{\sigma^2}} Ln(c_1(T-t,x,\alpha))c_1(T-t,x,\alpha) \\
- \; & e^{c(T-t)}\rho\nu\frac{4\alpha}{\sigma}\left(\frac{e^x}{L}\right)^{1-\frac{2\alpha}{\sigma^2}} \\
\times \; & \left\{C^{BS}\left(T-t,\frac{L^2}{e^x},\alpha,\sigma\right)\left\{1+(x-\log L)\left(1-\frac{2\alpha}{\sigma^2}\right)\right\}\right. \\
& \left. -(x-\log L)e^{-q(T-t)}\frac{L^2}{e^x}N(c_1(T-t,x,\alpha))\right\} \\
+ \; & \lambda(\theta-\sigma)e^{\alpha(T-t)}e^x n(d_1(T-t,x,\alpha))\sqrt{T-t} \\
- \; & \lambda(\theta-\sigma)\left(\frac{e^x}{L}\right)^{-\frac{2\alpha}{\sigma^2}} e^{\alpha(T-t)}Ln(c_1(T-t,x,\alpha))\sqrt{T-t} \\
- \; & e^{c(T-t)}\lambda(\theta-\sigma)\frac{4\alpha}{\sigma^3}\left(\log\frac{e^x}{L}\right)\left(\frac{e^x}{L}\right)^{1-\frac{2\alpha}{\sigma^2}} C^{BS}\left(T-t,\frac{L^2}{e^x},\alpha,\sigma\right),
\end{aligned} \tag{3.13}$$

and

$$c_1(t,x,\alpha) \;=\; \frac{2l-x-\log K+\alpha t}{\sigma\sqrt{t}}+\frac{1}{2}\sigma\sqrt{t}.$$

Proof. Firstly, note that when $k=1$ in Theorem 4, we have

$$v_1^0(T-t,x) \;=\; \int_0^t P_{t-r}^D\tilde{\mathscr{L}}_1^0 P_r^D f(x)dr.$$

Thus, we see the expansion

$$C_{\text{Barrier}}^{SV,\varepsilon}(T-t,e^x) = C_{\text{Barrier}}^{BS}(T-t,e^x,\alpha,\sigma,L)+\varepsilon\int_0^{T-t} P_s^D\tilde{\mathscr{L}}_1^0 P_{T-t-s}^D\bar{f}(x)ds+O(\varepsilon^2). \tag{3.14}$$

The first-order approximation term $v_1^0(t,x)=\displaystyle\int_0^{T-t} P_s^D\tilde{\mathscr{L}}_1^0 P_{T-t-s}^D\bar{f}(x)ds$ is given by

$$v_1^0(t,x) \;=\; \int_0^{T-t} e^{-cs}\bar{P}_s^D\tilde{\mathscr{L}}_1^0 e^{-c(T-t-s)}\bar{P}_{T-t-s}^D\bar{f}(x)ds$$

$$= e^{-c(T-t)} \int_0^{T-t} \bar{P}_s^D \tilde{\mathscr{L}}_1^0 \bar{P}_{T-t-s}^D \bar{f}(x) ds,$$

where $\bar{P}_t^D$ is defined by

$$\bar{P}_t^D \bar{f}(x) = \int_l^\infty \frac{1}{\sqrt{2\pi\sigma^2 s}} (1 - e^{-\frac{2(x-l)(y-l)}{\sigma^2 s}}) e^{-\frac{(y-x-(\alpha-\frac{1}{2}\sigma^2)s)^2}{2\sigma^2 s}} \bar{f}(y) dy.$$

Define $\vartheta(t,x)$ as

$$\begin{aligned}\vartheta(t,x) &= \tilde{\mathscr{L}}_1^0 \bar{P}_{T-t}^D f(e^x) \\ &= e^{c(T-t)} \rho\nu\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} C_{\text{Barrier}}^{BS}(T-t, e^x, \alpha, \sigma, L) + e^{c(T-t)} \lambda(\theta - \sigma) \frac{\partial}{\partial \sigma} C_{\text{Barrier}}^{BS}(T-t, e^x, \alpha, \sigma, L).\end{aligned}$$

A straightforward calculation shows that the above function agrees with the right-hand side of (3.13). Then we get the assertion. ■

Remark that through numerical integrations with respect to time s and space y in (3.12), we easily obtain the first order approximation of the down-and-out option prices.

Next, as a special case of (3.1) we consider the following stochastic volatility model with no drifts:

$$\begin{aligned} dS_t^\varepsilon &= \sigma_t^\varepsilon S_t^\varepsilon dB_t^1, \quad S_0^\varepsilon = S > 0, \\ d\sigma_t^\varepsilon &= \varepsilon\nu\sigma_t^\varepsilon (\rho dB_t^1 + \sqrt{1-\rho^2} dB_t^2), \quad \sigma_0^\varepsilon = \sigma > 0. \end{aligned} \tag{3.15}$$

where $\varepsilon \in [0,1)$, $\rho \in [-1,1]$ and $B = (B^1, B^2)$ is a two dimensional Brownian motion. In this case, we can provide a simpler approximation formula than in Theorem 5.

By Itô's formula, the following logarithmic model is obtained.

$$\begin{aligned} dX_t^\varepsilon &= -\frac{1}{2}(\sigma_t^\varepsilon)^2 dt + \sigma_t^\varepsilon dB_t^1, \quad X_0^\varepsilon = x = \log S, \\ d\sigma_t^\varepsilon &= \varepsilon\nu\sigma_t^\varepsilon (\rho dB_t^1 + \sqrt{1-\rho^2} dB_t^2), \quad \sigma_0^\varepsilon = \sigma. \end{aligned} \tag{3.16}$$

Again, the barrier option price is given by

$$C_{\text{Barrier}}^{SV,\varepsilon}(T, e^x) = \mathrm{E}\left[f(S_T^\varepsilon) 1_{\{\min_{0 \le u \le T} S_u^\varepsilon > L\}} \right],$$

where f stands for a payoff function and $L(< S)$ is a barrier price.

The differential operators $\mathscr{L}^\varepsilon$, $\tilde{\mathscr{L}}_1^0$ and the PDE are same as (3.3)–(3.5) with $c = q = 0$ and $\lambda = 0$. Also, the barrier option price in the Black-Scholes model coincides with (3.7) with no drift, that is,

$$C_{\text{Barrier}}^{BS}(T,S) = C^{BS}(T,S) - \left(\frac{S}{L}\right) C^{BS}\left(T, \frac{L^2}{S}\right),$$

where $C^{BS}(T,S)$ is the driftless Black-Scholes formula of the European call option given by

$$C^{BS}(T,S) = SN(d_1(T, \log S)) - KN(d_2(T, \log S))$$

with

$$
\begin{aligned}
d_1(t,x) &= d_1(t,x,0) = \frac{x - \log K + \sigma^2 t/2}{\sigma\sqrt{t}}, \\
d_2(t,x) &= d_2(t,x,0) = d_1(t,x) - \sigma\sqrt{t}.
\end{aligned}
$$

Then, we reach the following expansion formula which only needs 1-dimensional numerical integration.

Theorem 6. $C^{SV,\varepsilon}_{\text{Barrier}}(T,e^x) = C^{BS}_{\text{Barrier}}(T,e^x) + \varepsilon v_1^0(0,x) + O(\varepsilon^2)$, *where*

$$
\begin{aligned}
v_1^0(0,x) &= -\frac{1}{2}T\nu\rho\sigma\left\{e^x n(d_1(T,x))d_2(T,x) + Ln(c_1(T,x))c_1(T,x)\right\} \\
&\quad + \frac{\nu\rho L(x-l)\log(L/K)}{2\pi\sigma}\int_0^T \frac{(T-s)^{1/2}}{s^{3/2}}\exp\left(-\frac{c_2(T-s,L/K)+c_2(s,L/e^x)}{2}\right)ds, \\
c_1(t,x) &= \frac{\log(L^2/e^x K) + \sigma^2 t/2}{\sigma\sqrt{t}}, \quad c_2(t,y) = \left(\frac{\log y + \sigma^2 t/2}{\sigma\sqrt{t}}\right)^2. \qquad (3.17)
\end{aligned}
$$

Proof. See Appendix 6.4. ■

3.2 Numerical Example

Finally, applying the our approximation formulas in Theorem 5 and Theorem 6, we present numerical experiments for European down-and-out barrier call prices. First, let us denote $u^0 = C^{BS}_{\text{Barrier}}(T,S)$ and $v_1^0 = v_1^0(0,\log S)$. Then, we see

$$
C^{SV,\varepsilon}_{\text{Barrier}}(T,S) \simeq u^0 + \varepsilon v_1^0.
$$

In the following we report the results of the numerical experiments, where the numbers in the parentheses show the error rates (%) relative to the benchmark prices of $C^{SV,\varepsilon}_{\text{Barrier}}(T,S)$; they are computed by Monte-Carlo simulations with 100,000 time steps ($n = 10^5$) and 1,000,000 trials. We note that in our experiments the standard deviations of the benchmark Monte-Carlo simulations are calculated as at most 0.006 with the order of discretization error being 0.002 ($\approx \sqrt{T/n} = \sqrt{0.5/10^5}$ as stated in Introduction).

We check the accuracy of our approximations by changing the model parameters. Case 1–6 show the results for the stochastic volatility model with drifts of the underlying price process or/and the volatility process (3.1), while Case 7 shows the result for the stochastic volatility model with no drifts (3.15). There, we apply the formula in Theorem 5 to Case 1–6 and the formula in Theorem 6 to Case 7, respectively.

In all the cases, we set the initial asset price $S = 100$, the initial volatility $\sigma = 0.15$, the time to maturity $T = 0.5$, the lower barrier $L = 95$ and strike prices $K = 100, 102, 105$. The other parameters (c, q, $\varepsilon\nu$, ρ, $\varepsilon\lambda$, θ) are listed in the caption of each table.

Apparently, our approximation formula $u^0 + \varepsilon v_1^0$ improves the accuracy against $C^{SV,\varepsilon}_{\text{Barrier}}(T,S)$, and it is observed that εv_1^0 accurately compensates for the difference between $C^{SV,\varepsilon}_{\text{Barrier}}(T,S)$ and $C^{BS}_{\text{Barrier}}(T,S)$, which confirms the validity of our method.

Table 1: Case 1 ($c = 0.01$, $q = 0.0$, $\varepsilon\nu = 0.2$, $\rho = -0.5$, $\varepsilon\lambda = 0.0$, $\theta = 0.0$)

Strike	Benchmark	Our Approximation ($u^0 + \varepsilon v_1^0$)	Barrier Black-Scholes (u^0)
100	3.468	3.466 (-0.05%)	3.495 (0.80%)
102	2.822	2.822 (0.00%)	2.866 (1.57%)
105	1.986	1.986 (0.01%)	2.052 (3.36%)

Table 2: Case 2 ($c = 0.01$, $q = 0.0$, $\varepsilon\nu = 0.35$, $\rho = -0.7$, $\varepsilon\lambda = 0.0$, $\theta = 0.0$)

Strike	Benchmark	Our Approximation ($u^0 + \varepsilon v_1^0$)	Barrier Black-Scholes (u^0)
100	3.421	3.423 (0.07%)	3.495 (2.18%)
102	2.753	2.757 (0.18%)	2.866 (4.13%)
105	1.885	1.890 (0.23%)	2.052 (8.88%)

Table 3: Case 3 ($c = 0.05$, $q = 0.0$, $\varepsilon\nu = 0.35$, $\rho = -0.7$, $\varepsilon\lambda = 0.0$, $\theta = 0.0$)

Strike	Benchmark	Our Approximation ($u^0 + \varepsilon v_1^0$)	Barrier Black-Scholes (u^0)
100	4.352	4.349 (-0.07%)	4.399 (1.06%)
102	3.585	3.586 (0.02%)	3.665 (2.24%)
105	2.560	2.563 (0.11%)	2.696 (5.31%)

Table 4: Case 4 ($c = 0.05$, $q = 0.1$, $\varepsilon\nu = 0.2$, $\rho = -0.5$, $\varepsilon\lambda = 0.0$, $\theta = 0.0$)

Strike	Benchmark	Our Approximation ($u^0 + \varepsilon v_1^0$)	Barrier Black-Scholes (u^0)
100	2.231	2.224 (-0.31%)	2.268 (1.64%)
102	1.758	1.754 (-0.27%)	1.812 (3.02%)
105	1.172	1.168 (-0.31%)	1.243 (6.05%)

Table 5: Case 5 ($c = 0.01$, $q = 0.0$, $\varepsilon\nu = 0.2$, $\rho = -0.5$, $\varepsilon\lambda = 0.2$, $\theta = 0.25$)

Strike	Benchmark	Our Approximation ($u^0 + \varepsilon v_1^0$)	Barrier Black-Scholes (u^0)
100	3.523	3.517 (-0.16%)	3.495 (-0.77%)
102	2.891	2.888 (-0.09%)	2.866 (-0.85%)
105	2.066	2.065 (-0.06%)	2.052 (-0.64%)

Table 6: Case 6 ($c = 0.01$, $q = 0.0$, $\varepsilon\nu = 0.2$, $\rho = -0.5$, $\varepsilon\lambda = 0.5$, $\theta = 0.25$)

Strike	Benchmark	Our Approximation ($u^0 + \varepsilon v_1^0$)	Barrier Black-Scholes (u^0)
100	3.587	3.594 (0.20%)	3.495 (-2.55%)
102	2.976	2.987 (0.39%)	2.866 (-3.68%)
105	2.170	2.183 (0.59%)	2.052 (-5.41%)

Table 7: Case 7 ($c = 0.0$, $q = 0.0$, $\varepsilon\nu = 0.2$, $\rho = -0.5$, $\varepsilon\lambda = 0.0$, $\theta = 0.0$)

Strike	Benchmark	Our Approximation ($u^0 + \varepsilon v_1^0$)	Barrier Black-Scholes (u^0)
100	3.261	3.258 (-0.09%)	3.290 (0.90%)
102	2.640	2.639 (-0.02%)	2.686 (1.78%)
105	1.841	1.841 (0.01%)	1.911 (3.77%)

4 Conclusion

This paper has proposed an approximation scheme for barrier option prices by applying a new semi-group expansion to the Cauchy-Dirichlet problem in the second order parabolic partial differential equations (PDEs). As an application, we have derived a semi-group expansion formula under a certain type of stochastic volatility model and confirmed the validity of our method through numerical examples. Developing concrete computational schemes under various models is our next research topic.

5 Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] Crandall, M.G., H.Ishii and P.L.Lions (1993), "User's guide to viscosity solutions of second order partial differential equations", *Bull. A.M.S.*, **27**, 1–67.

[2] Fouque, J. P., G. Papanicolaou and K.R.Sircar (2000), "Mean Reverting Stochastic Volatility," *International Journal of Theoretical and Applied Finance*, **3**, 101-142.

[3] Fouque, J. P., G. Papanicolaou and K.R.Sircar (2000), *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, Cambridge.

[4] Fournie, E., J.-M.Lasry, J.Lebuchoux and P.-L.Lions (2001), "Applications of Malliavin calculus to Monte-Carlo methods in Finance II", *Finance Stoch.* **5**(2), 201–236.

[5] Friedman, A. (1964), *Partial differential equations of parabolic type*, Prentice-Hall.

[6] Friedman, A. (1975), *Stochastic differential equations and applications Volume 1*, Academic Press.

[7] Gobet, E. (2000), "Weak approximation of killed diffusion using Euler schemes", *Stochastic Processes and their Applications*, **87**, 167-197.

[8] Howison, E and M.Steinberg (2007), "A matched asymptotic expansions approach to continuity corrections for discretely sampled options. Part 1: Barrier options", Applied Mathematical Finance **14**, 63–89.

[9] Ilhan, A., M.Jonsson and K.R.Sircar (2004), "Singular Perturbations for Boundary Value Problems arising from Exotic Options", *SIAM Journal on Applied Mathematics*, **64** 1268-1293.

[10] Karatzas, I. and Shreve, S.E. (1991), Brownian Motion and Stochastic Calculus 2nd. edition., Springer, New York.

[11] Krylov, N.V. (1980), *Controlled diffusion processes,* Springer-Verlag Berlin.

[12] Kunitomo, N. and A.Takahashi (2003), "On validity of the asymptotic expansion approach in contingent claim analysis", *Annals of Applied Probability*, **13**, 914-952, 2003.

[13] Lamberton, D. and B.Lapeyre (1996), *Introduction to stochastic calculus applied to finance*, Chapman & Hall/CRC (translated by N.Rabeau and F.Mantion)

[14] Lieberman, G. M. (1996), *Second order parabolic differential equations*, World Scientific, River Edge, NJ, USA.

[15] Merton, R. C. (1973), *Theory of rational option pricing*, Bell Journal of Economics and Management Science, 4 (1), 141-183.

[16] Nagai, H. (1999), *Stochastic differential equations*, Kyoritsu Shuppan. (in Japanese)

[17] Pham, H. (2010), " Large deviations in mathematical finance", arXiv.

[18] Rubio, G. (2011), "The Cauchy-Dirichlet problem for a class of linear parabolic differential equations with unbounded coefficients in an unbounded domain", *International Journal of Stochastic Analysis,* 2011, Article ID 469806.

[19] Shiraya, K., A.Takahashi and T.Yamada. (2012), "Pricing discrete barrier options under stochastic volatility", *Asia-Pacific Financial Markets*, Volume 19, Issue 3, pp 205-232.

[20] Takahashi, A. and T. Yamada (2012), "An asymptotic expansion with push-down of Malliavin weights", *SIAM journal on Financial Mathematics*, **3**, 95-136.

6 Appendix

6.1 Proof of Theorem 2

First, by the definition of v^0_k, we easily get $v^0_k(T,x) = 0$ for $x \in D$ and $v^0_k(t,x) = 0$ for $(t,x) \in [0,T] \times \partial D$.

Next, fix any $x \in D$. By the Markov property, we have

$$\begin{aligned} J(t \wedge \tau_D(X^{0,x}))v^0_k\left(t \wedge \tau_D(X^{0,x}), X^{0,x}_{t\wedge\tau_D(X^{0,x})}\right) &= J(t)v^0_k(t, X^{0,x}_t)1_{\{\tau_D(X^{0,x})\geq t\}} \\ = \mathrm{E}\left[\int_t^{T\wedge\tau_D(X^{0,x})} J(r)g^0_k(r, X^{0,x}_r)dr \;\middle|\; \mathcal{F}_t\right] 1_{\{\tau_D(X^{0,x})\geq t\}} \\ = \mathrm{E}\left[\int_0^{T\wedge\tau_D(X^{0,x})} J(r)g^0_k(r, X^{0,x}_r)dr \;\middle|\; \mathcal{F}_t\right] - \int_0^{t\wedge\tau_D(X^{0,x})} J(r)g^0_k(r, X^{0,x}_r)dr \end{aligned}$$

for each $t \in [0,T]$, where $J(r) = \exp\left(-\int_0^r c(X^{0,x}_v, 0)dv\right)$ and $(\mathcal{F}_r)_r$ is the Brownian filtration. This implies that

$$M_t := J(t \wedge \tau_D(X^{0,x}))v^0_k\left(t \wedge \tau_D(X^{0,x}), X^{0,x}_{t\wedge\tau_D(X^{0,x})}\right) + \int_0^{t\wedge\tau_D(X^{0,x})} J(r)g^0_k(r, X^{0,x}_r)dr$$

is a local martingale. On the other hand, applying Ito's formula, we have that

$$\begin{aligned} M_t &= M_0 + \int_0^t \left\{ \left(\frac{\partial}{\partial t} + \mathscr{L}^0\right) v^0_k(r, X^{0,x}_r) + g^0_k(r, X^{0,x}_r) \right\} 1_{\{\tau_D(X^{0,x})\geq r\}}dr \\ &\quad + \sum_{i,j=1}^d \int_0^t J(r)\sigma^{ij}(X^{0,x}_r, 0)\frac{\partial}{\partial x^i}v^0_k(r, X^{0,x}_r)1_{\{\tau_D(X^{0,x})\geq r\}}dB^j_r \end{aligned}$$

for each $t \in [0,T]$. Thus, the uniqueness of decompositions of semimartingales gives us

$$\int_0^t \left\{ \left(\frac{\partial}{\partial t} + \mathscr{L}^0\right) v^0_k(r, X^{0,x}_r) + g^0_k(r, X^{0,x}_r) \right\} 1_{\{\tau_D(X^{0,x})\geq r\}}dr = 0, \quad t \in [0,T].$$

Therefore, for each fixed $t \in (0,T)$,

$$\frac{1}{h}\int_t^{t+h} \left\{ \left(\frac{\partial}{\partial t} + \mathscr{L}^0\right) v^0_k(r, X^{0,x}_r) + g^0_k(r, X^{0,x}_r) \right\} 1_{\{\tau_D(X^{0,x})\geq r\}}dr = 0$$

holds for any small enough $h > 0$. Since $x \in D$, by letting $h \to 0$, we obtain

$$\left(\frac{\partial}{\partial t} + \mathscr{L}^0\right) v_k^0(t,x) + g_k^0(t,x) = 0.$$

Finally we prove (2.9) by mathematical induction. When $k = 0$, the assertion is easily obtained by (2.12), (2.8), [F] and [G]. Now we assume that (2.9) holds for $1, 2, \ldots, k-1$. Then, by (2.8), (2.7) and [F], we have

$$|g_k^0(t,x)| \le C(1+|x|^{2m}) \sum_{|\alpha|\le 2} \left(|D_\alpha u^0(t,x)| + \sum_{l=1}^{k-1} |D_\alpha v_l^0(t,x)| \right)$$

for some $C, m > 0$, where $\alpha = (i_1, \ldots, i_d) \in \{0,1,2,\ldots\}^d$ is a multi-index, $|\alpha| = i_1 + \cdots + i_d$ and $D_\alpha = \partial^{|\alpha|}/(\partial x^1)^{i_1} \cdots (\partial x^d)^{i_d}$. By the induction hypothesis and [G]–[H], we see that

$$\sum_{|\alpha|\le 2} \left(|D_\alpha u^0(t,x)| + \sum_{l=1}^{k-1} |D_\alpha v_l^0(t,x)| \right) \le C' M(t)(1+|x|^{2m'})$$

for some $C', m' > 0$ and $M \in L^1([0,T).dt)$. Therefore, we get

$$|g_k^0(t,x)| \le C'' M(t)(1+|x|^{2m''})$$

for some $C'', m'' > 0$. Then we obtain

$$\begin{aligned}
|v_k^0(t,x)| &\le C'' \,\mathrm{E}\left[\int_0^{(T-t)} M(t+r)(1+|X_r^{0,x}|^{2m''})dr \right] \\
&\le C'' \left(1 + \mathrm{E}[\sup_{0\le r\le T} |X_r^{0,x}|^{2m''}]\right) \int_0^T M(r)dr \\
&\le C'' C_{m''} T^{m''-1} \left(\int_0^T M(r)dr \right) (1+|x|^{2m''})
\end{aligned}$$

by virtue of (2.12). Thus (2.9) also holds for k. Now we complete the proof of Theorem 2.

6.2 Proof of Theorem 3

First, we generalize the definitions of $\tilde{\mathscr{L}}_k^0$, g_k^0 and v_k^0. For $k, n \ge 1$, We define

$$\begin{aligned}
\tilde{\mathscr{L}}_k^\varepsilon &= \frac{1}{(k-1)!}\left\{ \frac{1}{2}\sum_{i,j=1}^d \int_0^1 (1-r)^{k-1} \frac{\partial^k a^{ij}}{\partial \varepsilon^k}(x, r\varepsilon)dr \frac{\partial^2}{\partial x^i \partial x^j} \right. \\
&\qquad \left. + \sum_{i=1}^d \int_0^1 (1-r)^{k-1} \frac{\partial^k b^i}{\partial \varepsilon^k}(x, r\varepsilon)dr \frac{\partial}{\partial x^i} - \int_0^1 (1-r)^{k-1}\frac{\partial^k c}{\partial \varepsilon^k}(x, r\varepsilon)dr \right\}, \\
g_n^\varepsilon(t,x) &= \tilde{\mathscr{L}}_n^\varepsilon u^0(t,x) + \sum_{k=1}^{n-1} \tilde{\mathscr{L}}_{n-k}^0 v_k^0(t,x) + \sum_{k=1}^{n-2} \varepsilon^k \left\{ \tilde{\mathscr{L}}_n^\varepsilon v_k^0(t,x) + \sum_{l=k+1}^{n-1} \tilde{\mathscr{L}}_{n+k-l}^0 v_l^0(t,x) \right\}
\end{aligned}$$

$$+\varepsilon^{n-1}\tilde{\mathscr{L}}_n^\varepsilon v_{n-1}^0(t,x),$$

where $g_1^\varepsilon(t,x)$ and $g_2^\varepsilon(t,x)$ are understood as $g_1^\varepsilon(t,x) = \tilde{\mathscr{L}}_1^\varepsilon u^0(t,x)$ and $g_2^\varepsilon(t,x) = \tilde{\mathscr{L}}_2^\varepsilon u^0(t,x) + \tilde{\mathscr{L}}_1^0 v_1^0(t,x) + \varepsilon\tilde{\mathscr{L}}_2^\varepsilon v_1^0(t,x)$, respectively.

We consider the following Cauchy-Dirichlet problem:

$$\begin{cases} -\dfrac{\partial}{\partial t}v(t,x) - \mathscr{L}^\varepsilon v(t,x) - g_n^\varepsilon(t,x) = 0, & (t,x) \in [0,T) \times D, \\ v(T,x) = 0, & x \in D, \\ v(t,x) = 0, & (t,x) \in [0,T] \times \partial D. \end{cases} \tag{6.1}$$

For $\varepsilon \neq 0$, we define $v_n^\varepsilon = [u^\varepsilon - \{u^0 + \sum_{k=1}^{n-1} \varepsilon^k v_k^0(t,x)\}]/\varepsilon^n$. Obviously, we see

$$u^\varepsilon(t,x) = u^0(t,x) + \sum_{k=1}^{n-1} \varepsilon^k v_k^0(t,x) + \varepsilon^n v_n^\varepsilon(t,x). \tag{6.2}$$

Proposition 3. *The function v_n^ε is a solution of* (6.1).

Proof. It is obvious that $v_n^\varepsilon(T,x) = 0$ for $x \in D$ and $v_n^\varepsilon(t,x) = 0$ for $(t,x) \in [0,T] \times \partial D$.

We also recall in (2.3) that

$$\frac{\partial}{\partial t}u^\varepsilon(t,x) + \mathscr{L}^\varepsilon u^\varepsilon(t,x) = 0.$$

Next, we apply Taylor's theorem to $\mathscr{L}^\varepsilon$ to observe that

$$\mathscr{L}^\varepsilon u^\varepsilon(t,x) = \left\{ \mathscr{L}^0 + \sum_{k=1}^{n-1} \varepsilon^k \tilde{\mathscr{L}}_k^0 + \varepsilon^n \tilde{\mathscr{L}}_n^\varepsilon \right\} u^\varepsilon(t,x). \tag{6.3}$$

Since u^0 is the solution of (2.3) with $\varepsilon = 0$, we get

$$\frac{\partial}{\partial t}u^0(t,x) + \mathscr{L}^0 u^0(t,x) = 0. \tag{6.4}$$

Similarly, since v_k^0 is a solution of (2.6), we have

$$\frac{\partial}{\partial t}v_k^0(t,x) + \mathscr{L}^0 v_k^0(t,x) + \tilde{\mathscr{L}}_k^0 u^0(t,x) + \sum_{l=1}^{k-1} \tilde{\mathscr{L}}_{k-l}^0 v_l^0(t,x) = 0. \tag{6.5}$$

Then, replacing $u^\varepsilon(t,x)$ in the right hand side of (6.3) by the right hand side of (6.2) and taking (6.4) and (6.5) into account, we obtain

$$\varepsilon^n \left\{ \frac{\partial}{\partial t}v_n^\varepsilon(t,x) + \mathscr{L}^0 v_n^\varepsilon(t,x) + \tilde{\mathscr{L}}_n^\varepsilon u^0(t,x) + \sum_{l=1}^{n-1} \tilde{\mathscr{L}}_{n-l}^0 v_l^0(t,x) \right\}$$
$$+ \sum_{k=n+1}^{2n-2} \varepsilon^k \left\{ \tilde{\mathscr{L}}_{k-n}^0 v_n^\varepsilon(t,x) + \tilde{\mathscr{L}}_n^\varepsilon v_{k-n}^0(t,x) + \sum_{l=k-n+1}^{n-1} \tilde{\mathscr{L}}_{k-l}^0 v_l^0(t,x) \right\}$$

$$+\varepsilon^{2n-1}\left\{\tilde{\mathscr{L}}^0_{n-1}v^\varepsilon_n(t,x)+\tilde{\mathscr{L}}^\varepsilon_n v^0_{n-1}(t,x)\right\}+\varepsilon^{2n}\tilde{\mathscr{L}}^\varepsilon_n v^\varepsilon_n(t,x)=0.$$

Thus, with the definition of $g^\varepsilon_n(t,x)$ above, we have for $\varepsilon\neq 0$,

$$\frac{\partial}{\partial t}v^\varepsilon_n(t,x)+\mathscr{L}^\varepsilon v^\varepsilon_n(t,x)+g^\varepsilon_n(t,x)=0.$$

This implies the assertion. ■

Set

$$\tilde{v}^\varepsilon_n(t,x)=\mathrm{E}\left[\int_0^{\tau_D(X^{\varepsilon,x})\wedge(T-t)}\exp\left(-\int_0^r c(X^{\varepsilon,x}_v,\varepsilon)dv\right)g^\varepsilon_n(t+r,X^{\varepsilon,x}_r)dr\right].$$

By [G]–[H], we find that there are $C_n>0$, $\tilde{m}_n\in\mathbb{N}$ which are independent of ε and the function $M_n\in C([0,T))\cap L^1([0,T),dt)$ determined by $u^0,v^0_1,\dots,v^0_{n-1}$ such that

$$|g^\varepsilon_n(t,x)|\le C_nM_n(t)(1+|x|^{2\tilde{m}_n}). \tag{6.6}$$

The inequalities (2.12) and (6.6) imply

$$|\tilde{v}^\varepsilon_n(t,x)|\le C'_n\int_t^T M_n(r)dr(1+|x|^{2\tilde{m}_n}) \tag{6.7}$$

for some $C'_n>0$ which is also independent of ε.

Proposition 4. $v^\varepsilon_n=\tilde{v}^\varepsilon_n$.

Proof. The assertion is easily obtained by the similar argument to the one in Theorem 5.1.9 in [13]. ■

Proof of Theorem 3. By (6.2) and Proposition 4, we have $u^\varepsilon(t,x)-(u^0(t,x)+\sum_{k=1}^{n-1}\varepsilon^k v^0_k(t,x))=\varepsilon^n\tilde{v}^\varepsilon_n(t,x)$. Our assertion is now immediately obtained by the inequality (6.7). ■

6.3 Proof of Theorem 4

1. Firstly, let us consider the case for $k=1$. Let $g\in\mathcal{H}^{m,1}$. Observe that

$$\begin{aligned}&\int_0^{(T-t)\wedge\tau_D(X^{0,x})}\exp\left(-\int_0^r c(X^{0,x}_v,0)dv\right)g(t+r,X^{0,x}_r)dr\\ =&\int_0^{T-t}\exp\left(-\int_0^r c(X^{0,x}_v,0)dv\right)g(t+r,X^{0,x}_r)1_{\{\tau_D(X^{0,x})\ge r\}}dr,\end{aligned}$$

and we obtain

$$\mathrm{E}\left[\int_0^{(T-t)\wedge\tau_D(X^{0,x})}\exp\left(-\int_0^r c(X^{0,x}_v,0)dv\right)g(t+r,X^{0,x}_r)dr\right]$$

$$
\begin{aligned}
&= \int_0^{T-t} \mathrm{E}\left[\exp\left(-\int_0^r c(X_v^{0,x},0)dv\right) g(t+r, X_r^{0,x}) 1_{\{\tau_D(X^{0,x})\geq r\}}\right] dr \\
&= \int_0^{T-t} P_r^D g(t+r,\cdot)(x)dr.
\end{aligned}
$$

Thus, under the assumption [H], we see

$$
\begin{aligned}
v_1^0(T-t,x) &= \mathrm{E}\left[\int_0^t \exp\left(-\int_0^r c(X_v^{0,x},0)dv\right) g_1^0(T-t+r, X_r^{0,x}) 1_{\{\tau_D(X^{0,x})\geq r\}} dr\right] \\
&= \int_0^t P_r^D \tilde{\mathscr{L}}_1^0 u^0(T-t+r,\cdot)(x)dr \\
&= \int_0^t P_r^D \tilde{\mathscr{L}}_1^0 P_{t-r}^D f(x)dr = \int_0^t P_{t-r}^D \tilde{\mathscr{L}}_1^0 P_r^D f(x)dr. \qquad (6.8)
\end{aligned}
$$

Thus, we have the assertion for $k=1$.

2. If the assertion holds for $1,\ldots,k-1$, then

$$
v_k^0(T-t,x) = \int_0^t P_{t_0}^D\{\tilde{\mathscr{L}}_k^0 u^0 + \sum_{l=1}^{k-1} \tilde{\mathscr{L}}_{k-l}^0 v_l^0\}(T-t+t_0,\cdot)(x)dt_0
$$

$$
\begin{aligned}
&= \int_0^t P_{t-t_0}^D \tilde{\mathscr{L}}_k^0 P_{t_0}^D f(x)dt_0 \\
&\quad + \sum_{l=1}^{k-1}\sum_{m=1}^{l} \sum_{(\beta^i)_{i=1}^m\subset\mathbb{N}^m, \sum_i \beta^i = l} \int_0^t \int_0^{t_0}\int_0^{t_1}\cdots\int_0^{t_{l-1}} \\
&\qquad P_{t-t_0}^D \tilde{\mathscr{L}}_{k-l}^0 P_{t_0-t_1}^D \tilde{\mathscr{L}}_{\beta^1}^0 P_{t_1-t_2}^D \tilde{\mathscr{L}}_{\beta^2}^0 \cdots P_{t_{l-1}-t_l}^D \tilde{\mathscr{L}}_{\beta^l}^0 P_{t_l}^D f(x) dt_l\cdots dt_1 dt_0 \\
&= \int_0^t P_{t-t_0}^D \tilde{\mathscr{L}}_k^0 P_{t_0}^D f(x)dt_0 \\
&\quad + \sum_{l=2}^{k}\sum_{m=1}^{l} \sum_{(\beta^i)_{i=1}^m\subset\mathbb{N}^m, \sum_i \beta^i = k} \int_0^t \int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_{l-1}} \\
&\qquad P_{t-t_1}^D \tilde{\mathscr{L}}_{\beta^1}^0 P_{t_1-t_2}^D \tilde{\mathscr{L}}_{\beta^2}^0 P_{t_2-t_3}^D \tilde{\mathscr{L}}_{\beta^3}^0 \cdots P_{t_{l-1}-t_l}^D \tilde{\mathscr{L}}_{\beta^l}^0 P_{t_l}^D f(x) dt_l\cdots dt_1 \\
&= \sum_{l=1}^{k} \sum_{(\beta^i)_{i=1}^l\subset\mathbb{N}^l, \sum_i \beta^i = k} \int_0^t \int_0^{t_1}\cdots\int_0^{t_{l-1}} P_{t-t_1}^D \tilde{\mathscr{L}}_{\beta^1}^0 P_{t_1-t_2}^D \tilde{\mathscr{L}}_{\beta^2}^0 \cdots P_{t_{l-1}-t_l}^D \tilde{\mathscr{L}}_{\beta^l}^0 P_{t_l}^D f(x) dt_l\cdots dt_1.
\end{aligned}
$$

Thus, our assertion is also true for k. Then we complete the proof of Proposition 4 by mathematical induction.

6.4 Proof of Theorem 6

By the asymptotic expansion in Section 2 and Theorem 4 with $k=1$, we see that the expansion

$$
C_{\text{Barrier}}^{SV,\varepsilon}(T,e^x) = C_{\text{Barrier}}^{BS,\varepsilon}(T,e^x) + \varepsilon v_1^0(0,x) + O(\varepsilon^2)
$$

holds with

$$v_1^0(t,x) = \int_0^{T-t} P^D_{T-t-r}\tilde{\mathscr{L}}_1^0 P^D_r \bar{f}(x)dr. \tag{6.9}$$

Then, we have the following proposition for an expression of $v_1^0(0,x)$. The proof is given in Section 6.4.1.

Proposition 5.

$$v_1^0(0,x) = \frac{T}{2}\nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}P^D_T\bar{f}(x) - \frac{1}{2}\mathrm{E}[(T-\tau_D(X^{0,x}))\nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}P^D_{T-\tau_D(X^{0,x})}\bar{f}(l)1_{\{\tau_D(X^{0,x})<T\}}].$$

We remark that the expectation in the above equality can be represented as

$$\begin{aligned} &\frac{1}{2}\mathrm{E}[(T-\tau_D(X^{0,x}))\nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}P^D_{T-\tau_D(X^{0,x})}\bar{f}(l)1_{\{\tau_D(X^{0,x})<T\}}] \\ = &\int_0^T \frac{(T-s)}{2}\nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}P^D_{T-s}\bar{f}(l)h(s,x-l)ds, \end{aligned} \tag{6.10}$$

where $h(s,x-l)$ is the density function of the first hitting time to l defined by

$$h(s,x-l) = \frac{-(l-x)}{\sqrt{2\pi\sigma^2 s^3}}\exp\left(-\frac{\{l-x+\sigma^2 s/2\}^2}{2\sigma^2 s}\right). \tag{6.11}$$

Now we evaluate

$$\nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}P^D_t\bar{f}(x) = \nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}C^{BS}(t,e^x) - \nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}\left\{\left(\frac{e^x}{L}\right)C^{BS}\left(t,\frac{L^2}{e^x}\right)\right\}.$$

Note that

$$\frac{\partial}{\partial\sigma}C^{BS}(t,e^x) = e^x n(d_1(t,x))\sqrt{t}, \tag{6.12}$$

and

$$\frac{\partial}{\partial\sigma}\left\{\left(\frac{e^x}{L}\right)C^{BS}\left(t,\frac{L^2}{e^x}\right)\right\} = Ln(c_1(t,x))\sqrt{t}. \tag{6.13}$$

Then we have

$$\begin{aligned} \nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}C^{BS}(t,e^x) &= \nu\rho\sigma^2 e^x n(d_1(t,x))\sqrt{t}\left\{1-\frac{d_1(t,x)}{\sigma\sqrt{t}}\right\} \\ &= -\nu\rho\sigma e^x n(d_1(t,x))d_2(t,x) \end{aligned} \tag{6.14}$$

and

$$\nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}\left\{\left(\frac{e^x}{L}\right)C^{BS}\left(t,\frac{L^2}{e^x}\right)\right\} = \nu\rho\sigma Ln(c_1(t,x))c_1(t,x). \tag{6.15}$$

Combining (6.12), (6.14) and (6.15), we get

$$\nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}P_t^D\bar{f}(x) \;=\; \nu\rho\sigma\left\{e^x n(d_1(t,x))(-d_2(t,x)) - Ln(c_1(t,x))c_1(t,x)\right\}. \quad (6.16)$$

Substituting (6.16) into (6.10), we have

$$\begin{aligned}\nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}P_t^D\bar{f}(l) &= \nu\rho\sigma Ln(d_1(t,l))(-d_2(t,l)) - \rho\sigma Ln(c_1(t,l))c_1(t,l)\\ &= \nu\rho\sigma Ln(d_1(t,l))(-(d_1(t,l)+d_2(t,l)))\\ &= \nu\rho\sigma\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\{l-\log K+\frac{1}{2}\sigma^2 t\}^2}{2\sigma^2 t}\right)\left(\frac{-2(l-\log K)}{\sigma\sqrt{t}}\right).\end{aligned}$$

Thus we obtain

$$\begin{aligned}&-\frac{1}{2}\,\mathrm{E}[(T-\tau_D(X^{0,x}))\nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}P^D_{T-\tau_D(X^{0,x})}\bar{f}(l)1_{\{\tau_D(X^{0,x})<T\}}]\\ =\;& -\int_0^T\frac{(T-s)}{2}\nu\rho\sigma L\frac{1}{\sqrt{2\pi}}e^{-\frac{\left\{l-\log K+\frac{1}{2}\sigma^2(T-s)\right\}^2}{2\sigma^2(T-s)}}\left(\frac{-2(l-\log K)}{\sigma\sqrt{T-s}}\right)\\ &\times\frac{-(l-x)}{\sqrt{2\pi\sigma^2 s^3}}e^{-\frac{\left\{(l-x)+(\sigma^2/2)s\right\}^2}{2\sigma^2 s}}ds\\ =\;& \frac{\nu\rho L(x-l)\log(L/K)}{2\pi\sigma}\int_0^T\frac{(T-s)^{1/2}}{s^{3/2}}\exp\left(-\frac{c_2(T-s,L/K)+c_2(s,L/e^x)}{2}\right)ds.\end{aligned} \quad (6.17)$$

By Proposition 5, (6.10), (6.16) and (6.17), we reach the assertion.

6.4.1 Proof of Proposition 5

First, we notice the following relation:

$$\tilde{\mathscr{L}}_1^0 P_t^D\bar{f}(x) = \nu\rho\sigma^3 t\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)P_t^D\bar{f}(x). \quad (6.18)$$

Then, using the relations $\mathscr{L}^0\tilde{\mathscr{L}}_1^0 P_t^D\bar{f}(x) = \tilde{\mathscr{L}}_1^0\mathscr{L}^0 P_t^D\bar{f}(x)$ and

$$\left(\frac{\partial}{\partial t}+\mathscr{L}^0\right)P^D_{T-t}\bar{f}(x) = 0,$$

we get

$$\left(\frac{\partial}{\partial t}+\mathscr{L}^0\right)\frac{T-t}{2}\tilde{\mathscr{L}}_1^0 P^D_{T-t}\bar{f}(x) \;=\; -\tilde{\mathscr{L}}_1^0 P^D_{T-t}\bar{f}(x). \quad (6.19)$$

Also, we have

$$\left(\frac{\partial}{\partial t}+\mathscr{L}^0\right)\int_0^{T-t}P^D_{T-t-r}\left(\nu\rho\sigma^2\frac{\partial^2}{\partial x\partial\sigma}P_r^D\bar{f}\right)(x)dr = -\tilde{\mathscr{L}}_1^0 P^D_{T-t}\bar{f}(x),\quad x\in(l,\infty). \quad (6.20)$$

Therefore, the function

$$\eta(t,x) = \int_0^{T-t} P^D_{T-t-r}\left(\nu\rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P^D_r \bar{f}\right)(x)dr - \frac{T-t}{2}\tilde{\mathscr{L}}^0_1 P^D_{T-t}\bar{f}(x) \tag{6.21}$$

satisfies the following PDE

$$\begin{cases} \left(\frac{\partial}{\partial t} + \mathscr{L}^0\right)\eta(t,x) = 0, & (t,x) \in [0,T) \times (l,\infty), \\ \eta(T,x) = 0, & x \in [l,\infty), \\ \eta(t,l) = -\frac{T-t}{2}\tilde{\mathscr{L}}^0_1 P^D_{T-t}\bar{f}(l), & t \in [0,T). \end{cases}$$

Then Theorem 6.5.2 in [6] implies

$$\eta(0,x) = -\frac{1}{2}\,\mathrm{E}[(T-\tau_D(X^{0,x}))\nu\rho\sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P^D_{T-\tau_D(X^{0,x})}\bar{f}(l) 1_{\{\tau_D(X^{0,x})<T\}}]. \tag{6.22}$$

By (6.21) and (6.22), we get the assertion. ■