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Abstract

We investigate combinatorial auctions from a practical perspective. The auctioneer gathers information according to a dynamical protocol termed ask price procedure. We demonstrate a method for elucidating whether a procedure gathers sufficient information for deriving a VCG mechanism. We calculate representative valuation functions in a history-contingent manner, and show that it is necessary and sufficient to examine whether efficient allocations with and without any buyer associated with the profile of representative valuation functions were revealed. This method is tractable, and can be applied to general procedures with connectedness. The representative valuation functions could be the sufficient statistics for privacy preservation.

Keywords: Combinatorial Auctions, Ask Price Procedure, Price-Based VCG Mechanisms, Connectedness, Representative Valuation Functions.

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1. Introduction

This paper investigates the combinatorial auction problem, wherein a single seller sells multiple, indivisible items with multiple units to multiple buyers who have private and quasi-linear valuations; these items are divided into multiple disjoint packages in order to sell them in an efficient manner. We investigate general dynamical auction protocols in a continuous time horizon a la Tatonnement, which are termed *ask price procedures* in this paper's terminology. According to an ask price procedure, the auctioneer gathers information regarding the buyers' valuations on packages in that he (or she) continues to ask prices of each buyer and requires this buyer to announce a collection of packages as his (or her) demand correspondence. In order to consider the buyers' convenience, the auctioneer can restrict the range of each buyer's demand correspondences in a history-contingent manner.

This paper examines whether an arbitrary given ask price procedure can successfully gather sufficient information regarding the buyers' valuations to derive an efficient allocation. In this case, the usage of the gathered information must be compatible with the buyers' incentives in terms of voluntary participation and sincere demand correspondence. This paper also elucidates the extent to which the information regarding the buyers' valuations, which may be irrelevant to the auctioneer's decision, could be leaked to the public.

The revelation principle, addressed by Myerson (1979), implies that an arbitrary well-behaved indirect mechanism could be replaced with a direct mechanism, wherein each buyer is required to announce his entire valuations simultaneously on possible packages. Many researchers have intensively studied a special class of direct mechanism named *VCG (Vickrey-Clarke-Groves)* mechanisms[1], which are the only *efficient* mechanisms that are *strategy-proof*, that is, incentive compatible in terms of dominant strategies, and *ex post individually rational*, that is, never drive the buyers into deficit in ex post terms.[2]

The standard practice of VCG mechanisms, wherein buyers directly announce their

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[1] See Vickrey (1961), Clarke (1971), and Groves (1973).

[2] For the surveys on mechanism design in general, see, for example, Fudenberg and Tirole (1993, Chapter 7) and Mas-Colell, Whinston, and Green (1995, Chapter 23).

entire valuations, has serious flaws from a practical standpoint in terms of *complexity and privacy preservation*. First, since the number of possible packages is exponential with regard to the number of items, it may be too complicated for any buyer who has normal limitations on his cognitive ability to assess and report his entire valuations on all packages simultaneously. Second, any buyer may be concerned about preserving his privacy, because he is afraid that any information that is confidential but not relevant to the auctioneer's decisions could be leaked to the public. The revelation principle, however, does not address these concerns.[3] Hence, it would be very meaningful for the combinatorial auction problem to attempt to search for the possibility of replacing the standard practice of a direct mechanism with an alternative indirect auction protocol, in order to make information gathering compatible with the addressing of practical issues such as complexity and privacy preservation.

Market design approaches, including those of Kelso and Crawford (1982), Gul and Stacchetti (2000), Ausubel (2006), Ausubel, Cramton, and Milgrom (2006), Parkes and Ungar (2002), Lahaie and Parkes (2004), and Mishra and Parkes (2007), have examined various concepts for dynamical combinatorial auction design to address the dilemma over complexity and privacy preservation. It is a well-accepted view in the communication complexity literature that the replacement of a simultaneous direct revelation with a dynamical protocol that involves price adjustments including feedback could drastically reduce the communication cost.[4] Moreover, we could imagine that any popular protocol is generally characterized by some tacit limitations concerning the extent to which the auctioneer can make ask prices non-linear and non-unanimous and the extent to which he can limit the range of the packages from which each buyer can select his demand correspondence. Hence, an important research direction as the alternative to a more ideal protocol design would be to explore generally applicable and tractable calculation methods for elucidating whether an arbitrary given ask price procedure can gather sufficient information for deriving a VCG mechanism. In other words, the research will elucidate whether there exists a VCG mechanism that is

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[3] For the criticisms of VCG mechanisms or direct mechanisms in general, see, for example, Rothkopf, Teisberg, and Kahn (1990), Milgrom (2004), Ausubel and Milgrom (2006), Parkes (2006), and Segal (2006).

[4] See, for example, Nisan and Segal (2006) and Segal (2006), for example.

price-based on the ask price procedure according to this paper's terminology, and how this VCG mechanism can explicitly be derived.

This paper introduces an assumption on an ask price procedure named *connectedness*, implying that the auctioneer cannot make his ask prices jump discontinuously to prices that he has never asked before, as well as implying that the auctioneer cannot entirely prohibit any buyer from revealing the same package as the one that he has revealed before. This connectedness has a nice property in that irrespective of the fine detail of protocol specifications, the auctioneer can always identify the difference in valuation between packages that a buyer has revealed. This property would drastically simplify the identification of the side payments induced by a VCG mechanism. Despite this nice property, the set of connected ask price procedures are still too extensive to cover all the concepts of dynamical protocol design discussed in the combinatorial auction literature, such as the multiple linear price trajectories addressed by Ausubel (2006), and the primal-dual algorithm discovering a universal competitive equilibrium addressed by Parkes and Ungar (2002) and Mishra and Parkes (2007).

The main contribution of this paper is to demonstrate the following general calculation method for elucidating whether there exists a VCG mechanism that is price-based on an arbitrary given connected ask price procedure, and for deriving such a VCG mechanism. On the basis of the history regarding the sequence of ask prices and demand correspondences, we can define the *representative valuation function* for each buyer by assigning any revealed package with the minimal relative valuations in a manner that is consistent with the history. The representative valuation function exists uniquely and can be easily calculated in a history-contingent manner. The auctioneer can at any time, irrespective of what has occurred in the history, easily identify whether he (or she) has succeeded in gathering sufficient information for implementing a VCG outcome using this calculated profile of representative valuation functions alone. All the auctioneer has to do for this identification is to examine whether history reveals the efficient allocation and the efficient allocations without any single buyer associated with the profile of representative valuation functions.

There are many researches, including Kelso and Crawford (1982), Gul and Stacchetti (2000), Bikhchandani and Ostroy (2002), Parkes and Ungar (2002), Lahaie and Parkes (2004), and Mishra and Parkes (2007), that investigated the relationship between the possibility of a mechanism being price-based on an ask price procedure and the discovery of competitive equilibrium prices in this procedure. In particular, Parkes and Ungar (2002), Lahaie and Parkes (2004), and Mishra and Parkes (2007) introduced an involved notion termed *universal competitive equilibrium*, and showed an important characterization implying that the auctioneer can gather sufficient information to implement a VCG mechanism if and only if the gathered information identifies a universal competitive equilibrium. However, these works failed to provide any method for elucidating which universal competitive equilibrium could be identified. They also failed to clarify the extent to which the information regarding the buyers' valuations could be leaked.

In contrast to these works, the present paper successfully provides such a method, because the profile of representative valuation functions could be a universal competitive equilibrium. Importantly, because of the connectedness, we can show that the profile of representative valuation functions, along with the sets of all revealed packages, could be the sufficient statistics for privacy preservation, that is, the extent to which the information regarding the buyers' valuations could be leaked.

On the basis of this paper's main contribution, we can argue regarding more constructive aspects of price-based VCG mechanisms. Assume that, according to an ask price procedure, when an auctioneer never stops the asking of prices, he can eventually make the buyers reveal all packages; the auctioneer surely identifies a time at which the associated profile of representative valuation functions implies a universal competitive equilibrium in this case. Then, by stopping the asking of prices at this time, the auctioneer can successfully implement a VCG outcome.

Finally, from the viewpoint of the incomplete contract literature, such as Tirole (1999), we will argue with regard to the possibility that the auctioneer's discretion to select an ask price procedure at the pre-play stage further preserves the buyers' privacy. With this discretion, the auctioneer can utilize his private information regarding the buyers' valuations. In this case, since the profile of representative valuation functions and the sets of all revealed packages can verify whether his selected outcome is VCG, the auctioneer does not have to make any agreement beforehand with the buyers and the sellers regarding the fine details of the procedure.

In order to prevent the buyers from behaving strategically and promote their meaningful biddings, activity rules were proposed by several authors and incorporated into real situations. For example, see Milgrom (2004) and Ausubel, Cramton, and Milgrom (2006). Accordingly, the present paper assumes the *revealed preference* activity rule in the sense that throughout the ask price procedure, any buyer is required to make his demand responses consistent with a single valuation function.

Several previous works, including Kelso and Crawford (1982), Gul and Stacchetti (2000), and Ausubel (2006), have imposed restrictions on the buyers' valuations such as substitute conditions. In contrast to these works, the present paper does not put any such restriction on valuations; we admit all cases that include mixtures of substitutes and complements. Moreover, many works have focused on ascending auction protocols. In contrast, this paper considers general price adjustments including ascending types, descending types, or a mixture of ascending and descending.

The remainder of this paper is organized as follows: Section 2 models the combinatorial auction problem. Section 3 introduces the concepts of ask price procedure and price-based mechanism. Section 4 introduces the concept of connectedness. Section 5 introduces the concept of representative valuation functions. Section 6 shows a necessary and sufficient condition under which there exists a VCG mechanism that is price-based on an arbitrary given ask price procedure. Section 7 shows the main contribution of this paper, that is, demonstrating a calculation method according to which the necessary and sufficient condition in Section 6 can be replaced with a much more tractable condition using representative valuation functions. Section 8 investigates any ask price procedure, wherein the buyers reveal all packages in the long run, provided the auctioneer never stops asking prices. Section 9 clarifies the conceptual relationship between representative valuation functions and universal competitive equilibrium. Section 10 discusses the role of the auctioneer's discretion. Section 11 concludes this paper.

2. Model

Let us investigate a combinatorial auction problem wherein $l \geq 1$ multiple items exist, with m_z multiple units for each item $z \in \{1,...,l\}$ that a single seller supplies to multiple buyers. The set of buyers is denoted by $N \equiv \{1,...,n\}$. A *package for each buyer* $i \in N$ is denoted by $a_i = (a_{i1},...,a_{il})$, where $a_{iz} \in \{0,...,m_z\}$ denotes the amount of item z for buyer i. Let $A_i \equiv \prod_{z\in\{1,...,l\}} \{0,...,m_z\}$. Let $\underline{a}_i \in A_i$ denote the null package for buyer i, where

$$\underline{a}_{iz} = 0 \text{ for all } z \in \{1,...,l\}.$$

An *allocation* is denoted by $a \equiv (a_1,...,a_n) \in \prod_{i\in N} A_i$, where

$$\sum_{i\in N} a_{iz} \leq m_z \text{ for each } z \in \{1,...,l\}.$$

Let $A \subset \times_{i\in N} A_i$ denote the set of all allocations. *An allocation without a buyer* $i \in N$ is defined as $a_{-i} \equiv (a_j)_{j\in N\setminus\{i\}}$, where

$$\sum_{j\in N\setminus\{i\}} a_{jz} \leq m_z \text{ for each item } z \in \{1,...,l\}.$$

Let $A_{-i}^i \subset \prod_{j\in N\setminus\{i\}} A_j$ denote the set of all allocations without a buyer $i \in N$.

A *valuation function for buyer* $i \in N$ is defined as $u_i : A_i \to R$; where it is quasi-linear, $u_i(\underline{a}_i) = 0$, and any increase in the amount of items has a positive value; for every $\{a_i, a_i'\} \subset A_i$

(1) $$u_i(a_i) > u_i(\tilde{a}_i) \text{ if } a_i \neq \tilde{a}_i \text{ and } a_i \geq \tilde{a}_i.$$

We do not impose any further restriction on the possible valuation functions; we consider any type that mixes substitutes and complements. Let U_i denote the set of all valuation functions for buyer i. Let $U \equiv \prod_{i\in N} U_i$, $U_{-i} \equiv \prod_{j\in N\setminus\{i\}} U_j$, $u = (u_i)_{i\in N} \in U$, and $u_{-i} = (u_j)_{j\in N\setminus\{i\}} \in U_{-i}$. An allocation $a \in A$ is said to be *efficient for* $u \in U$ if

$$\sum_{i\in N} u_i(a_i) \geq \sum_{i\in N} u_i(\tilde{a}_i) \text{ for all } \tilde{a} \in A.$$

Let $A^*(u) \subset A$ denote the set of all efficient allocations for $u \in U$. An allocation $a_{-i} \in A_{-i}^i$ without a buyer i is said to be *efficient for* $u_{-i} \in U_{-i}$ if

$$\sum_{j \in N \setminus \{i\}} u_j(a_i) \geq \sum_{j \in N \setminus \{i\}} u_j(\tilde{a}_i) \quad \text{for all} \quad \tilde{a}_{-i} \in A_{-i}^i .$$

Let $A_{-i}^{i*}(u_{-i}) \subset A_{-i}^i$ denote the set of all efficient allocations without buyer i for $u_{-i} \in U_{-i}$.

A *direct mechanism*, hereinafter a *mechanism*, is defined as $G = (g, q)$, where $g: U \to A$ denotes the allocation function, and $q: U \to R^n$ denotes the side payment function. Let us denote $g(u) = (g_i(u))_{i \in N} \in A$, $q = (q_i)_{i \in N}$, $q_i: U \to R$, and $q(u) = (q_i(u))_{i \in N} \in R^n$. When the auctioneer applies the mechanism G to the combinatorial auction problem in the standard manner, and those players having a true profile of valuation functions $u \in U$ directly announce that their profile of valuation functions is $\tilde{u} \in U$, the auctioneer will select the allocation $g(\tilde{u}) \in A$ and make the side payment $q_i(\tilde{u}) \in R$ from each buyer $i \in N$ to the seller; the resulting payoff for buyer i is given by

$$u_i(g_i(\tilde{u})) - q_i(\tilde{u}),$$

and the resulting payoff for the seller is given by

$$\sum_{i \in N} q_i(\tilde{u}) .$$

A mechanism G is said to be *efficient* if

$$g(u) \in A^*(u) \quad \text{for all} \quad u \in U .$$

A mechanism G is said to be *strategy-proof* if truth telling is a dominant strategy at all times, that is, for every $i \in N$, every $u \in U$, and every $\tilde{u}_i \in U_i$,

$$u_i(g(u)) - q_i(u) \geq u_i(g(\tilde{u}_i, u_{-i})) - q_i(\tilde{u}_i, u_{-i}) .$$

A mechanism G is said to be *ex post individually rational* if the seller and buyers have incentive to participate in the combinatorial auction problem, that is, for every $u \in U$,

$$\sum_{i \in N} q_i(\tilde{u}) \geq 0 ,$$

and

$$u_i(g(u)) - q_i(u) \geq 0 \quad \text{for all} \quad i \in N .$$

It is a well-accepted view that a mechanism is efficient, strategy-proof, and ex post individually rational if and only if it is a *VCG* mechanism in the following sense.

Definition 1: A mechanism G is *VCG* if for every $u \in U$,

$$g(u) \in A^*(u),$$

and

$$q_i(u) = \max_{a_{-i} \in A_{-i}^i} \sum_{j \in N \setminus \{i\}} u_j(a_j) - \sum_{j \in N \setminus \{i\}} u_j(g_j(u)) \quad \text{for all} \quad i \in N.$$

We shall focus on VCG mechanisms throughout this paper.

3. Price-Based Mechanisms

A *price vector for buyer* $i \in N$ is denoted by $p_i = (p_i(a_i))_{a_i \in A_i} \in R^{|A_i|}$, where we assumed that $p_i(\underline{a}_i) = 0$, and

(2) $$p_i(a_i) > p_i(\tilde{a}_i) \text{ if } a_i \neq \tilde{a}_i \text{ and } a_i \geq \tilde{a}_i .$$

Let P_i denote the set of all price vectors for buyer i. Let $p \equiv (p_i)_{i \in N} \in \prod_{i \in N} P_i$ denote a profile of price vectors; we consider the possibility that prices are non-linear and non-anonymous.

Let

$$E_i \equiv \{M_i \subset A_i \mid M_i \neq \phi\}$$

denote the set of all non-empty subsets of packages for buyer i. We consider the situation wherein instead of applying the standard practice in which the buyers directly announce their valuation functions, the auctioneer will apply the following dynamical auction protocol. At any time t in the continuous time horizon $[0, \infty)$, the auctioneer asks a price vector $p_i(t) \in P_i$ of each buyer $i \in N$, and he (or she) requires this buyer to announce his (or her) demand correspondence as the collection of the best response packages among a restricted subset of packages given by $M_i(t) \in E_i$. Correspondingly, this buyer announces his demand correspondence as a non-empty subset, that is, $m_i(t) \subset M_i(t)$.

A combination of a price vector, a subset of packages, and a demand correspondence for buyer i, which is denoted by $(p_i, M_i, m_i) \in P_i \times E_i \times E_i$, where it was assumed that $m_i \subset M_i$, is said to be *consistent with* a valuation function $u_i \in U_i$ for buyer i if m_i is equivalent to the set of all best responses to p_i with a restriction on M_i, that is,

$$m_i = \arg\max_{a_i \in M_i} \{u_i(a_i) - p_i(a_i)\} .$$

A *history for each buyer,* $i \in N$ *, up to each time,* $t \in (0, \infty)$, is denoted by

$$h_i^t : [0, t) \to P_i \times E_i \times E_i ,$$

where, for each $\tau \in [0,t)$, we denoted $h_i^t(\tau) = (p_i(\tau), M_i(\tau), m_i(\tau))$ and assumed $m_i(\tau) \subset M_i(\tau)$. It is said to be *consistent with* $u_i \in U_i$ if $h_i^t(\tau) \in P_i \times E_i \times E_i$ is consistent with u_i for each $\tau \in [0,t)$. Let h_i^0 denote the *null* history. Let $H_i^t(u_i)$ denote the set of all histories for buyer i up to time t that is consistent with u_i. Let $H_i^t \equiv \bigcup_{u_i \in U_i} H_i^t(u_i)$, $H^t \equiv \prod_{i\in N} H_i^t$, $H \equiv \bigcup_{t\in[0,\infty)} H^t$, $h^t = (h_i^t)_{i\in N} \in H^t$, and $H_i^0 \equiv \{h_i^0\}$. For every $h_i^t \in H_i^t$, we define the set of all valuation functions for buyer i with which h_i^t is consistent as

$$U_i(h_i^t) \equiv \{u_i \in U_i \mid h_i^t \in H_i^t(u_i)\}.$$

For every $h_i^t \in H_i^t$, let

$$A_i(h_i^t) \equiv \{a_i \in A_i \mid a_i \in m_i(\tau) \ \textit{for some}\ \tau \in [0,t)\}.$$

This denotes the set of all packages for buyer i that he announces as his demand response in the history h_i^t, where $h_i^t(\tau) = (p_i(\tau), M_i(\tau), m_i(\tau))$ for each $\tau \in [0,t)$. Let $A(h^t) \equiv \prod_{i\in N} A_i(h_i^t)$.

An *ask price procedure*, describing the practice of a dynamical auction protocol, is defined as (γ, λ, T), where $\gamma = (\gamma_i)_{i\in N}$ denotes the *price adjustment rule*, $\gamma_i : H \to P_i$ for each $i \in N$, and $\lambda = (\lambda_i)_{i\in N}$ denotes the *demand restriction rule*, $\lambda_i : H \to E_i$ for each $i \in N$, and $T : U \to (0,\infty)$ denotes the *stopping time rule*. According to the price adjustment rule γ, at any time $t \in [0,\infty)$ where $h^t \in H^t$ has occurred, the auctioneer asks the price vector $\gamma_i(h^t) \in P_i$ of each buyer $i \in N$. In this case, according to the demand restriction rule λ, the auctioneer restricts this buyer's demand response packages to the subset $\lambda_i(h^t) \subset A_i$. For the sake of convenience, we assume that at the initial time 0, the auctioneer does not restrict each buyer's demand responses:

$$\lambda_i(h^0) = A_i \quad \text{for all} \quad i \in N.$$

Let $h^t = h^t(u, \gamma, \lambda)$ denote the history up to time t that occurs when the buyers continue to announce their demand correspondences in the manner consistent with the profile of valuation functions u and the auctioneer continues to ask price vectors according to the price adjustment rule γ and the demand restriction rule λ:

$$h^t(u,\gamma,\lambda) \in H^t(u),$$

and

$$p_i(\tau) = \gamma_i(h^\tau) \text{ and } M_i(\tau) = \lambda_i(h^\tau) \text{ for all } i \in N \text{ and all } \tau \in [0,t),$$

where $h^t(u,\gamma,\lambda) = (h_i^t(u,\gamma,\lambda))_{i \in N}$, and for every $i \in N$,

$$h_i^t(u,\gamma,\lambda)(\tau) = h_i^t(\tau) = (p_i(\tau), M_i(\tau), m_i(\tau)) \text{ for all } \tau \in [0,t).$$

In order to maintain the generality of the argument, we permit non-linearity and non-anonymity of price asking. However, it must be noted that it is implicitly assumed that a particular specification of a price adjustment rule γ reflects some practical obstacle that prevents the auctioneer to a greater or lesser extent from having a free hand to manage the ask price adjustment.

Importantly, we assume the *revealed preference* activity rule in the sense that throughout the ask price procedure, any buyer is required to make his demand responses consistent with a single valuation function. When the auctioneer follows the price adjustment rule γ and the demand restriction rule λ, and the buyers continue to make their demand correspondence in a manner consistent with the profile of valuation functions $u \in U$, the auctioneer stops asking price vectors at the time given by $t = T(u) \in (0,\infty)$. In this case, it should be assumed that the stopping time rule T is contingent only on the history: for every $\{u,u'\} \subset U$, if $h^{T(u)}(u,\gamma,\lambda)$ is consistent with u', that is, if $u' \in U(h^{T(u)}(u,\gamma,\lambda))$, then

$$T(u') = T(u), \text{ and } h^{T(u)}(u,\gamma,\lambda) = h^{T(u')}(u',\gamma,\lambda)$$

must hold. Let

$$H(\gamma,\lambda,T) \equiv \{h^t \in H \mid h^t = h^{T(u)}(u,\gamma,\lambda) \textit{ for some } u \in U\}.$$

This denotes the set of all histories that can occur under the assumption of revealed preference activity rule.

Definition 2: A mechanism $G = (g,q)$ is *price-based on an ask price procedur*e (γ,λ,T), if for every $h^t \in H(\gamma,\lambda,T)$ and every $\{u,u'\} \subset U(h^t)$,

$$(g(u),q(u)) = (g(u'),q(u')).$$

The price-based property in Definition 2 implies that the auctioneer's selection of an allocation and side payments is contingent only on the public information about the profile of valuation functions that the auctioneer collects in the ask price procedure (γ,λ,T).[5] The following lemma shows that *whenever a mechanism is efficient and price-based on an ask price procedure, the allocation induced by the mechanism must be revealed by the buyers in this ask price procedure*.

Lemma 1: *If a mechanism* G *is efficient and price-based on an ask price procedure* (γ,λ,T)*, then*

$$g(u)\in A(h^{T(u)}(u,\gamma,\lambda)) \quad \textit{for all} \quad u\in U\,.$$

Proof: For every $\varepsilon>0$ and every $u\in U$, we define $u_{i,\varepsilon}\in U_i$ as

$$u_{i,\varepsilon}(\underline{a}_i)=0\,,$$

and

$$u_{i,\varepsilon}(a_i)=u_i(a_i)+\varepsilon \quad \text{for all} \quad a_i\in A_i\setminus\{\underline{a}_i\}\,.$$

Assume that there exists $u\in U$ and $i\in N$ such that

$$g_i(u)\notin A_i(h_i^{T(u)}(u,\gamma,\lambda))\,.$$

Note that if

$$g_i(u)=\underline{a}_i\notin A_i(h_i^{T(u)}(u,\gamma,\lambda))\,,$$

then, $h_i^{T(u)}(u,\gamma,\lambda)$ must also be consistent with $u_{i,\varepsilon}$ for all $\varepsilon>0$, that is,

$$u_{i,\varepsilon}\in U_i(h_i^{T(u)}(\gamma,\lambda,T)) \quad \text{for all} \quad \varepsilon>0\,,$$

which, along with the price-based property and efficiency of G, implies that

$$g(u_{i,\varepsilon},u_{-i})=g(u)\in A^*(u_{i,\varepsilon},u_{-i}) \quad \text{for all} \quad \varepsilon>0\,.$$

This is, however, a contradiction, because any efficient allocation, namely, $a\in A^*(u_{i,\varepsilon},u_{-i})$ for $(u_{i,\varepsilon},u_{-i})$, never satisfies $a_i\neq\underline{a}_i$, provided ε is sufficiently large.

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[5] We address this point in subsection 9.2.

Assume that

$$g_i(u) \neq \underline{a}_i \, .$$

Then, we can select $a_i \in A_i \setminus \{g_i(u)\}$ such that

$$a_i \leq g_i(u) \, ,$$

and for every $a_i' \in A_i \setminus \{a_i, g_i(u)\}$ such that $a_i' \leq g_i(u)$,

$$u_i(g_i(u)) - u_i(a_i) \leq u_i(g_i(u)) - u_i(a_i') \, .$$

From inequalities (1), we can select $u_i' \neq u_i$ with which $h_i^{T(u)}(\gamma, \lambda, T)$ is consistent, that is,

$$u_i' \in U_i(h_i^{T(u)}(\gamma, \lambda, T)) \, ;$$

we can also select $j \in N \setminus \{i\}$ in a way that $u_i'(g_i(u)) - u_i'(a_i)$ is close enough to zero to satisfy the following equation:

(3) $$u_i'(g_i(u)) - u_i'(a_i) < u_j(g_j(u) + g_i(u) - a_i) - u_j(g_j(u)) \, .$$

Let us specify $\hat{a} \in A$ by

$$\hat{a}_i = a_i \, ,$$

$$\hat{a}_j = g_j(u) + g_i(u) - a_i \, ,$$

and

$$\hat{a}_h = g_h(u) \quad \text{for all} \quad h \in N \setminus \{i, j\} \, .$$

From inequality (3),

$$u_i'(g_i(u)) + \sum_{h \in N \setminus \{i\}} u_h(g_h(u)) < u_i'(\hat{a}_i) + \sum_{h \in N \setminus \{i\}} u_h(\hat{a}_h) \, ,$$

which implies that $g(u)$ is not efficient for (u_i', u_{-i}). However, since $h_i^{T(u)}(u, \gamma, \lambda)$ is consistent with u_i', it must hold that $g(u_i', u_{-i}) = g(u)$. This contradicts the efficiency of G.

Q.E.D.

4. Connectedness

A history h_i^t for buyer i up to time t is said to be *connected* if for every $\tau \in (0,t)$, either

$$p_i(\tau) = \lim_{\tau' \uparrow \tau} p_i(\tau'),\quad M_i(\tau) \subset \lim_{\tau' \uparrow \tau} M_i(\tau'),\text{ and }\quad M_i(\tau) \cap \lim_{\tau' \uparrow \tau} m_i(\tau') \neq \phi,$$

or, there exists $\tau' \in (0,\tau)$ such that

$$p_i(\tau) = p_i(\tau'),\quad M_i(\tau) \subset M_i(\tau'),\text{ and }\quad M_i(\tau) \cap m_i(\tau') \neq \phi,$$

where $h_i^t(\tau) = (p_i(\tau), M_i(\tau), m_i(\tau))$. The connectedness implies that the auctioneer *never* makes his asked price vector *jump* discontinuously to any price vector that he has never asked before. It also implies that the auctioneer never prohibits the possibility that a buyer announces the same package as the one that he has revealed before. Moreover, it implies that whenever the auctioneer permits a buyer to reveal a package that this buyer has been prohibited from revealing, he must go back to a previous time at which this buyer was permitted to reveal this package.[6] This last aspect of the implication indicates that this paper accepts the concept of multiple linear price trajectories addressed by Ausubel (2006).

The following lemma shows that on the assumption of this connectedness, *the auctioneer can calculate the difference in valuation for any buyer between any pair of packages whenever this buyer reveals these packages in the ask price procedure.*

Lemma 2: *For every history,* $h_i^t \in H_i^t$*, that is connected, and for every* $\{a_i, a_i'\} \subset A_i(h_i^t)$*, there uniquely exists* $x_i(a_i, a_i', h_i^t) \in R$ *such that for every* $u_i \in U_i(h_i^t)$*,*

$$x_i(a_i, a_i', h_i^t) = u_i(a_i) - u_i(a_i').$$

Proof: Since $h_i^t \in H_i^t$ is connected, there exists a finite sequence of times and actions $(\tau^l, a_i^l)_{l=1}^k$ such that

k is a positive integer that is greater than 1,

[6] It is possible to go back to such a time because we assumed that $\lambda_i(h^0) = A_i$ for all $i \in N$.

$$a_i^1 = a_i',$$

$$a_i^k = a_i,$$

$$\tau^l \in [0,t) \text{ and } a_i^l \in m_i(\tau^l) \text{ for all } l \in \{1,...,k\},$$

and

$$a_i^{l-1} \in m_i(\tau^l) \text{ for all } l \in \{2,...,k\},$$

where $h_i^t(\tau) = (p_i(\tau), M_i(\tau), m_i(\tau))$. For every $u_i \in U_i(h_i^t)$ and every $l \in \{2,...,k\}$, since $\{a_i^l, a_i^{l-1}\} \subset m_i(\tau^l)$, it follows that

$$u_i(a_i^l) - p_i(\tau^l)(a_i^l) = u_i(a_i^{l-1}) - p_i(\tau^l)(a_i^{l-1}).$$

Hence,

(4) $$u_i(a_i) - u_i(a_i') = \sum_{l=2}^{k} \{u_i(a_i^l) - u_i(a_i^{l-1})\} = \sum_{l=2}^{k} \{p_i(\tau^l)(a_i^l) - p_i(\tau^l)(a_i^{l-1})\}.$$

Let us specify $x_i(a_i, a_i', h_i^t) \in R$ as

$$x_i(a_i, a_i', h_i^t) = \sum_{l=2}^{k} \{p_i(\tau^l)(a_i^l) - p_i(\tau^l)(a_i^{l-1})\}.$$

Since this specification does not depend on the selection of $u_i \in U_i(h_i^t)$, it is clear from the equalities (4) that for every $u_i \in U_i(h_i^t)$ and every $\{a_i, a_i'\} \subset A_i(h_i^t)$,

$$u_i(a_i) - u_i(a_i') = x_i(a_i, a_i', h_i^t).$$

Q.E.D.

An ask price procedure (γ, λ, T) is said to be *connected* if for every $t \in (0, \infty)$, every $u \in U$, and every $i \in N$, $h_i^t(u, \gamma, \lambda) \in H_i^t$ is connected. Throughout this paper, we shall confine our attention to ask price procedures that are connected. Because of Lemma 2, the concept of connectedness along with that of representative valuations, which will be addressed in the next section, will play the central role in calculating the side payments induced by a VCG mechanism. Despite this nice property, the class of connected ask price procedures is still too extensive to include various concepts for combinatorial auction design, such as the multiple linear price trajectories addressed by Ausubel (2006) and the primal-dual algorithm for discovering a universal competitive equilibrium, addressed by Parkes and Ungar (2002) and Mishra and Parkes (2004). This paper considers general price adjustments including ascending types, descending types, or mixtures of ascending and descending.

5. Representative Valuation Functions

For every buyer $i \in N$, every time $t \in (0,\infty)$, and every history $h_i^t \in H_i^t$ that is connected, we define the *representative valuation function,* $u_i^{[h_i^t]} \in U_i$ according to the following tractable method of calculation, which should be regarded as this paper's key concept, together with connectedness. Assume $u_i^{[h_i^t]}(\underline{a}_i) = 0$, and fix an arbitrary package for buyer i that belongs to $A_i(h_i^t)$, which is denoted by $\tilde{a}_i \in A_i(h_i^t)$. For every $a_i \in A_i(h_i^t) \setminus \{\tilde{a}_i\}$, specify

$$u_i^{[h_i^t]}(a_i) \equiv u_i^{[h_i^t]}(\tilde{a}_i) - x_i(\tilde{a}_i, a_i, h_i^t),$$

and for every $a_i \notin A_i(h_i^t)$,

$$u_i^{[h_i^t]}(a_i) \equiv \inf_{\substack{\tau \in [0,t), a_i \in M_i(\tau), \\ a_i' \in m_i(\tau)}} \{u_i^{[h_i^t]}(a_i') - p_i(\tau)(a_i') + p_i(\tau)(a_i)\},$$

where we denote that $h_i^t(\tau) = (p_i(\tau), M_i(\tau), m_i(\tau))$. The latter part of the specifications implies that the representative valuation function assigns the maximal absolute value to any *unrevealed* non-null package in a manner that is consistent with the history. Let $u^{[h^t]} = (u_i^{[h_i^t]})_{i \in N}$. Because $\lambda_i(h^0) = A_i$ was assumed, it is clear that the representative valuation function $u_i^{[h_i^t]}$ exists uniquely.

The following proposition shows that the representative valuation function $u_i^{[h_i^t]}$ assigns any *revealed* package $a_i \in A_i(h_i^t)$ with the *minimal* possible valuation in relative terms. It also shows that the set of all valuation functions consistent with the history $h_i^t \in H_i^t$, that is, $U_i(h_i^t)$, can be uniquely identified from the representative valuation function $u_i^{[h_i^t]}$ and the set of revealed packages $A_i(h_i^t)$. Hence, the representative valuation function $u_i^{[h_i^t]}$, along with the set of revealed packages $A_i(h_i^t)$, could be regarded as the *sufficient statistics* concerning the extent to which the information about buyer $i's$ valuation function u_i was leaked in the history h_i^t.

Proposition 3: *For every* $t \in [0,\infty)$, *every connected history* $h_i^t \in H_i^t$, *and every*

$u_i \in U_i$ *, it holds that* $u_i \in U_i(h_i^t)$ *, if and only if for every* $a_i \in A_i(h_i^t)$ *,*

$$u_i(a_i) - u_i(a_i') = u_i^{[h_i^t]}(a_i) - u_i^{[h_i^t]}(a_i') \quad \textit{for all} \quad a_i' \in A_i(h_i^t),$$

and

$$u_i(a_i) - u_i(a_i') > u_i^{[h_i^t]}(a_i) - u_i^{[h_i^t]}(a_i') \quad \textit{for all} \quad a_i' \notin A_i(h_i^t).$$

In this case,

$$u_i(a_i) \geq u_i^{[h_i^t]}(a_i),$$

and

$$u_i(a_i) = u_i^{[h_i^t]}(a_i) \quad \textit{if and only if} \quad \underline{a}_i \in A_i(h_i^t).$$

Proof: The proof of the "if" part is straightforward from the definition of $u_i^{[h_i^t]}$. From the definition of $u_i^{[h_i^t]}$ and $a_i \in A_i(h_i^t)$, if $u_i \in U_i(h_i^t)$, then for every $a_i' \in A_i$,

$$u_i(a_i) - u_i(a_i') \geq u_i^{[h_i^t]}(a_i) - u_i^{[h_i^t]}(a_i'),$$

and $a_i' \in A_i(h_i^t)$, if and only if

$$u_i(a_i) - u_i(a_i') = u_i^{[h_i^t]}(a_i) - u_i^{[h_i^t]}(a_i') = x_i(a_i, a_i', h_i^t),$$

where we have used the assumption of revealed preference activity rule and Lemma 2.

By letting $a_i' = \underline{a}_i$, from $u_i(\underline{a}_i) = u_i^{[h_i^t]}(\underline{a}_i) = 0$ and the above characterization that $u_i(a_i) \geq u_i^{[h_i^t]}(a_i)$, it follows that

$$u_i(a_i) = u_i^{[h_i^t]}(a_i) \quad \text{if and only if} \quad \underline{a}_i \in A_i(h_i^t).$$

Q.E.D.

6. Price-Based VCG Mechanisms

The following proposition shows a necessary and sufficient condition for the existence of a price-based VCG mechanism; it is necessary and sufficient that *associated with any profile of valuation functions, the efficient allocation and the efficient allocations without any single buyer are revealed in the ask price procedure.*

Proposition 4: *There exists a VCG mechanism* G *that is price-based on a connected ask price procedure* (γ,λ,T) *if and only if for every* $h^t \in H(\gamma,\lambda,T)$, *there exist* $a^*(h^t) \in A(h^t)$, *and* $a_{-i}^{i*}(h_{-i}^t) \in A_{-i}(h_{-i}^t)$ *for each* $i \in N$, *such that for every* $u \in U(h^t)$,

(5) $$a^*(h^t) \in A^*(u),$$

and

(6) $$a_{-i}^{i*}(h_{-i}^t) \in A_{-i}^{i*}(u_{-i}) \text{ for all } i \in N.$$

Proof: We prove the "if" part as follows. Assume that for every $h^t \in H(\gamma,\lambda,T)$, there exist $a^*(h^t) \in A(h^t)$, and $a_{-i}^{i*}(h_{-i}^t) \in A_{-i}(h_{-i}^t)$ for each $i \in N$, that satisfy the properties (5) and (6) for all $u \in U(h^t)$. Then, we can specify $g: U \to A$ in a way that for every $h^t \in H(\gamma,\lambda,T)$ and every $u \in U(h^t)$,

$$g(u) = a^*(h^t).$$

We can also specify $q_i : U \to R$ for each $i \in N$ in a way that for every $h^t \in H(\gamma,\lambda,T)$ and every $u \in U(h^t)$,

$$q_i(u) = \sum_{j \in N\setminus\{i\}} x_j(a_j^{i*}(h_{-i}^t), g_j(u), h^t).$$

From Lemma 2 and property (6), it follows that

$$q_i(u) = \max_{a_{-i} \in A_{-i}^i} \sum_{j \in N\setminus\{i\}} u_j(a_j) - \sum_{j \in N\setminus\{i\}} u_j(g_j(u)),$$

which along with property (5) implies that the correspondingly specified mechanism $G = (g,q)$ is VCG.

We prove the "only if" part as follows. Assume that $G = (g,(q_i)_{i \in N})$ is VCG and price-based on (γ,λ,T), where

$$g(u)\in A^{*}(u) \quad \text{for all} \quad u\in U .$$

Note from inequalities (1) and (2) that for every $i\in N$, every $h_i^t\in H_i^t$, and every $\{a_i,\tilde{a}_i\}\not\subset A_i(h_i^t)$, there exists $\{u_i,\tilde{u}_i\}\subset U_i(h_i^t)$ such that

$$u_i(a_i)-u_i(\tilde{a}_i)\neq \tilde{u}_i(a_i)-\tilde{u}_i(\tilde{a}_i) .$$

Hence, for every $u\in U$, if either

$$g(u)\notin A(h^{T(u)}(u,\gamma,\lambda))$$

or

$$A_{-j}(h^{T(u)}(u,\gamma,\lambda))\bigcap A_{-j}^{j*}(u_{-j})=\phi \quad \text{for some} \quad j\in N ,$$

then there exist $j\in N$ and $\tilde{u}_j\in U_j$ such that

$$(\tilde{u}_j,u_{-j})\in U(h^{T(u)}(u,\gamma,\lambda)) ,$$

and for every $i\in N\setminus\{j\}$,

$$q_i(\tilde{u}_j,u_{-j})=\max_{a_{-i}\in A_{-i}^{i}}\{\tilde{u}_j(a_j)+\sum_{h\in N\setminus\{i,j\}}u_h(a_h)\}-\{\tilde{u}_j(g_j(u))+\sum_{h\in N\setminus\{i,j\}}u_h(g_h(u))\}$$

$$\neq \max_{a_{-i}\in A_{-i}^{i}}\sum_{h\in N\setminus\{i\}}u_h(a_h)-\sum_{h\in N\setminus\{i\}}u_h(g_h(u))$$

$$=q_i(u) .$$

This contradicts the supposition that G is price-based on (γ,λ,T). Hence, we have proved that for every $u\in U$,

$$g(u)\in A(h^{T(u)}(u,\gamma,\lambda)) ,$$

and

$$A_{-j}(h^{T(u)}(u,\gamma,\lambda))\bigcap A_{-j}^{j*}(u_{-j})\neq\phi \quad \text{for all} \quad j\in N .$$

Assume that there exist $\{u,\tilde{u}\}\subset U$, $j\in N$, and $a_{-j}\in A_{-j}^{j}$ such that

$$\tilde{u}\in U(h^{T(u)}(u,\gamma,\lambda)) ,$$

$$a_{-j}\in A_{-j}(h_{-j}^{T(u)}(u,\gamma,\lambda)) ,$$

$$a_{-j}\in A_{-j}^{j*}(u_{-j}) ,$$

and

$$a_{-j}\notin A_{-j}^{j*}(\tilde{u}_{-j}) .$$

Without loss of generality, we can select $\tilde{u}$ satisfying that

$$q_j(\tilde{u}) = \max_{\tilde{a}_{-j} \in A_{-j}^j} \sum_{i \in N \setminus \{j\}} \tilde{u}_i(\tilde{a}_i) - \sum_{i \in N \setminus \{j\}} \tilde{u}_i(g_i(u))$$

$$> \sum_{i \in N \setminus \{j\}} \tilde{u}_i(a_i) - \sum_{i \in N \setminus \{j\}} \tilde{u}_i(g_i(u)).$$

Since

$$g(u) \in A(h^{T(u)}(u,\gamma,\lambda)) \quad \text{and} \quad a_{-j} \in A_{-j}(h_{-j}^{T(u)}(u,\gamma,\lambda)),$$

it follows that

$$\sum_{i \in N \setminus \{j\}} \tilde{u}_i(a_i) - \sum_{i \in N \setminus \{j\}} \tilde{u}_i(g_i(u)) = \sum_{i \in N \setminus \{j\}} u_i(a_i) - \sum_{i \in N \setminus \{j\}} u_i(g_i(u)) = q_i(u),$$

which implies that

$$q_i(\tilde{u}) \neq q_i(u).$$

This contradicts the assumption that G is price-based on (γ,λ,T). Hence, we have proved that for every $u \in U$ and every $j \in N$,

$$A_{-j}(h_{-j}^{T(u)}(u,\gamma,\lambda)) \cap \Big(\bigcap_{\tilde{u}_{-j} \in U_{-j}(h_{-j}^{T(u)}(u,\gamma,\lambda))} A_{-j}^{j*}(\tilde{u}_{-j}) \Big) \neq \phi,$$

which implies that there exists $a_{-i}^{i*}(h_{-i}^t) \in A_{-i}(h_{-i}^t)$ that satisfies property (6). Moreover, Lemma 1 implies that there exists $a^*(h^t) \in A(h^t)$ for each $i \in N$ satisfying property (5).

From the above observations, we have proved the “only if” part; properties (5) and (6) are necessary for the existence of a VCG mechanism that is price-based on (γ,λ,T).

Q.E.D.

From the connectedness and Lemma 2, it follows that whenever the auctioneer recognizes that an efficient allocation and an efficient allocation without each buyer were revealed, he can calculate the differences in valuation among these allocations, and therefore, can calculate the side payments induced by a VCG mechanism.

7. Main Theorem

The following theorem, which should be regarded as the main theorem, shows that the necessary and sufficient condition in Proposition 4 can be replaced with another condition, implying that associated with the profile of representative valuation functions, there exist an efficient allocation and an efficient allocation without a single buyer that are all revealed in the ask price procedure. This condition could be much simpler and more tractable, because all we have to do for evaluating the sufficiency of this condition is to *examine the case of representative valuation functions.*

Theorem 5: *There exists a VCG mechanism* G *that is price-based on a connected ask price procedure* (γ,λ,T), *if and only if for every* $h^t \in H(\gamma,\lambda,T)$,

(7) $$A(h^t)\bigcap A^*(u^{[h^t]}) \neq \phi,$$

and

(8) $$A_{-i}(h^t_{-i})\bigcap A^{i*}_{-i}(u^{[h^t_{-i}]}_{-i}) \neq \phi \quad \textit{for all} \quad i \in N.$$

Proof: From Proposition 3 and the specification of $u^{[h^t]}$, it follows that for every $i \in N$, every $a_i \in A_i(h^t_i)$, and every $\tilde{a}_i \in A_i$,

$$u^{[h^t_i]}_i(a_i) - u^{[h^t_i]}_i(\tilde{a}_i) \leq u_i(a_i) - u_i(\tilde{a}_i) \quad \text{for all} \quad u_i \in U_i(h^t_i).$$

Hence, for every $a \in A(h^t)$,

$$a \in A^*(u) \quad \text{for all} \quad u \in U(h^t) \quad \text{if} \quad a \in A^*(u^{[h^t]}).$$

From the specification of $u^{[h^t]}$, it follows that for every $a \in A(h^t)$,

$$a \in A^*(u^{[h^t]}) \quad \text{if} \quad a \in A^*(u) \quad \text{for all} \quad u \in U(h^t).$$

Hence, we have proved that for every $a \in A(h^t)$,

$$a \in A^*(u^{[h^t]}), \text{ if and only if } a \in A^*(u) \quad \text{for all} \quad u \in U(h^t).$$

This observation implies that property (7) is equivalent to property (5). In the same manner, we can also prove that for every $j \in N$ and every $a_{-j} \in A_{-j}(h^t_{-j})$,

$$a_{-j} \in A^{j*}_{-j}(u^{[h^t]}_{-j}), \text{ if and only if } a_{-j} \in A^{j*}_{-j}(u_{-j}) \quad \text{for all} \quad u_{-j} \in U_{-j}(h^t_{-j}).$$

This observation implies that property (8) is equivalent to property (6).

Q.E.D.

Since the profile of representative valuation functions $u^{[h^t]}$ minimizes the differences in valuation between the efficient allocations and other allocations, the requirements for efficiency would be the severest among all relevant profiles of valuation functions, $u \in U(h^t)$; hence, it is sufficient to just examine the profile of representative valuation functions in this case.

We should recall the implication of Proposition 3 that the set of all valuation functions consistent with $h^t \in H(\gamma, \lambda, T)$, that is, $U_i(h_i^t)$, can be uniquely identified from the representative valuation function $u_i^{[h_i^t]}$ and the set of revealed packages $A_i(h_i^t)$. Hence, the extent to which the information regarding the buyers' valuations is leaked to the public can be fully expressed by the profile of representative valuation functions and the set of revealed packages.

8. Sufficiency

We can give a practical interpretation of an ask price procedure as follows: At any time $t \in [0,t)$, where $h^t \in H^t$ has occurred, the auctioneer calculates the corresponding profile of representative valuation functions $u^{[h^t]}$, and examines whether there exist the associated efficient allocation $a \in A$ and efficient allocations $a_{-i}^i \in A_{-i}^i$ without any buyer, $i \in N$ that has been revealed. If the auctioneer identifies such allocations, he stops asking price vectors, selects the allocation a, and makes the side payment $s_i \in R$ from each buyer i to the seller, which is expressed as

$$s_i = \sum_{j \in N \setminus \{i\}} u_j(a_j^i) - \sum_{j \in N \setminus \{i\}} u_j(a_j).$$

If the auctioneer fails to find such allocations, he continues to ask price vectors until he can find them.

Using this interpretation, let us consider any price adjustment rule γ and demand restriction rule λ, according to which, each buyer can reveal *all* packages in the long run; it is evident from Theorem 5 that there exists a stopping time T and a VCG mechanism G such that G is price-based on the associated ask price procedure (γ, λ, T).

Proposition 6: *Assume that a combination of price adjustment rule and demand restriction rule* (γ, λ) *satisfy that*

$$\lim_{t \to \infty} A(h^t(u, \gamma, \lambda)) = A \quad \textit{for all} \quad u \in U.$$

Then, there exist a stopping time rule T *and a VCG mechanism* G *such that* G *is price-based on the associated ask price procedure* (γ, λ, T).

Proof: The proof is evident from Theorem 5.

Q.E.D.

We can show a sufficient condition for a combination (γ, λ) in order to satisfy the assumption in Proposition 6. Fix an arbitrary positive but small real number $\varepsilon > 0$.

Assume that a price adjustment rule γ and a demand restriction rule λ satisfy that for every $u \in U$, every $t \in (0,\infty)$, and every $i \in N$, either

$$A_i(h_i^t(u,\gamma,\lambda)) = A_i,$$

or, there exist $t' \in (t,\infty)$ and $a_i \notin A_i(h_i^t(u,\gamma,\lambda))$ such that

$$a_i \in M_i(t'),$$

and

$$u_i^{[h_i^t(u,\gamma,\lambda)]}(a_i) - u_i^{[h_i^t(u,\gamma,\lambda)]}(a_i') \geq p_i(t')(a_i) - p_i(t')(a_i') + \varepsilon,$$

where a_i' was a package that was revealed up to time t and is included in $M_i(t')$, that is,

$$a_i' \in A_i(h_i^t(u,\gamma,\lambda)) \cap M_i(t').$$

In this case, the difference in representative valuation between the unrevealed package a_i and the revealed package a_i' shrinks as time goes by:

$$u_i^{[h_i^t(u,\gamma,\lambda)]}(a_i) - u_i^{[h_i^t(u,\gamma,\lambda)]}(a_i') > u_i^{[h_i^{t'}(u,\gamma,\lambda)]}(a_i) - u_i^{[h_i^{t'}(u,\gamma,\lambda)]}(a_i').$$

In particular, whenever $a_i \in A_i(h_i^{t'}(u,\gamma))$, then it must hold that

$$u_i^{[h_i^t(u,\gamma,\lambda)]}(a_i) - u_i^{[h_i^t(u,\gamma,\lambda)]}(a_i') \geq u_i^{[h_i^{t'}(u,\gamma,\lambda)]}(a_i) - u_i^{[h_i^{t'}(u,\gamma,\lambda)]}(a_i') + \varepsilon.$$

Proposition 7: *If a price adjustment rule γ and a demand restriction rule λ satisfy the assumption of this section, then, for every $u \in U$, there exists $t \in (0,\infty)$ such that*

$$A_i(h_i^t(u,\gamma,\lambda)) = A_i \quad \textit{for all } i \in N.$$

Proof: Consider an arbitrary $u \in U$. Assume that there exist $i \in N$ and $a_i \in A_i$, such that

$$a_i \notin A_i(h_i^t(u,\gamma,\lambda)) \quad \text{for all } t \in [0,\infty).$$

In this case, without loss of generality, we can select such a package $a_i \in A_i$ and an infinite sequence of times $(t^s)_{s=1}^{\infty}$ such that

$$\lim_{s\to\infty} t^s = \infty,$$

and

$$u_i^{[h_i^{t^s}(u,\gamma,\lambda)]}(a_i) - u_i^{[h_i^{t^s}(u,\gamma,\lambda)]}(a_i') \geq u_i^{[h_i^{t^{s+1}}(u,\gamma,\lambda)]}(a_i) - u_i^{[h_i^{t^{s+1}}(u,\gamma,\lambda)]}(a_i') + \varepsilon \,,$$

where $a_i' \in A_i(h_i^t(u,\gamma,\lambda))$. This implies that the value of a_i diverges into infinity, that is,

$$\lim_{s\to\infty} u_i^{[h_i^{t^s}(u,\gamma,\lambda)]}(a_i) = \infty \,,$$

which is a contradiction.

Q.E.D.

9. Universal Competitive Equilibrium

This section will argue that the ask price procedure could be regarded as a price discovery process. Because of inequalities (1) and (2), we can express the representative valuation function by a $|A_i|$-dimensional vector as $u_i^{[h_i^t]} = (u_i^{[h_i^t]}(a_i))_{a_i \in A_i}$, which we could regard as a price vector for buyer i, that is, $u_i^{[h_i^t]} \in P_i$. This section will show that the profile of representative valuation functions could be regarded as a *universal competitive equilibrium*, the notion of which was addressed by Parkes and Ungar (2002) and Mishra and Parkes (2004).

A profile of price vectors, $p \equiv (p_i)_{i \in N} \in \prod_{i \in N} P_i$, is said to be a *competitive equilibrium for* $u \in U$, if there exists an allocation $a^{CE}(u) \in A$ that maximizes the payoffs for the seller and the buyers, that is,

$$\sum_{i \in N} p_i(a_i^{CE}) \geq \sum_{i \in N} p_i(a_i) \quad \text{for all} \quad a \in A,$$

and for every $i \in N$ and $a_i \in A_i$,

$$u_i(a_i^{CE}) - p_i(a_i^{CE}) \geq u_i(a_i) - p_i(a_i).$$

The concept of competitive equilibrium is an extension of Walrasian equilibrium to the combinatorial auction problem. A profile of price vectors, $p \in \prod_{i \in N} P_i$, is said to be a *competitive equilibrium without buyer* i *for* $u_{-i} \in U_{-i}$ if there exists an allocation without buyer i, $a_{-i}^{i,CE}(u) \in A_{-i}^i$, that maximizes the payoffs for the sellers and the buyers, except for buyer i, satisfying that

$$\sum_{j \in N \setminus \{i\}} p_j(a_j^{i,CE}) \geq \sum_{j \in N \setminus \{i\}} p_j(a_j) \quad \text{for all} \quad a_{-i} \in A_{-i}^i,$$

and for every $j \in N \setminus \{i\}$ and $a_j \in A_j$,

$$u_j(a_j^{i,CE}) - p_j(a_j^{i,CE}) \geq u_j(a_j) - p_j(a_j).$$

The concept of universal competitive equilibrium is defined as a profile of price vectors that satisfies both competitive equilibrium and competitive equilibrium without any buyer. A profile of price vectors, $p \in \prod_{i \in N} P_i$, is said to be a *universal competitive* *equilibrium for* $u \in U$ if it is a competitive equilibrium for u, and for every $i \in N$, it is a competitive equilibrium without buyer i for u_{-i}. Whenever p is a universal competitive equilibrium for u, then the allocations $a^{CE}(u)$ and $a_{-i}^{i,CE}(u)$ could satisfy the properties of efficiency in that

$$a^{CE}(u) \in A^*(u),$$

and

$$a_{-i}^{i,CE}(u) \in A_{-i}^{i*}(u_{-i}) \text{ for all } i \in N.$$

For a further discussion on these equilibrium concepts, see Parkes (2006).

Proposition 8: *For every* $h^t \in H^t$*, if properties (7) and (8) are satisfied, then the profile of representative valuation functions* $u_i^{[h_i^t]}$ *is a universal competitive equilibrium for all* $u \in U(h_i^t)$.

Proof: From properties (7) and (8), we can select $a^*(h^t) \in A(h^t)$, and $a_{-i}^{i*}(h_{-i}^t) \in A_{-i}(h_{-i}^t)$ for each $i \in N$, such that

$$a^*(h^t) \in A(h^t) \cap A^*(u^{[h^t]}),$$

and

$$a_{-i}^{i*}(h_{-i}^t) \in A_{-i}(h_{-i}^t) \cap A_{-i}^{i*}(u_{-i}^{[h_{-i}^t]}) \text{ for all } i \in N.$$

Hence,

$$\sum_{i \in N} u_i^{[h_i^t]}(a_i^*(h_i^t)) \geq \sum_{i \in N} u_i^{[h_i^t]}(a_i) \text{ for all } a \in A,$$

and for every $i \in N$,

$$\sum_{j \in N \setminus \{i\}} u_j^{[h_j^t]}(a_i^{j*}(h_i^t)) \geq \sum_{j \in N \setminus \{i\}} u_j^{[h_j^t]}(a_j) \text{ for all } a_{-i} \in A_{-i}^i.$$

From Proposition 3, $a^*(h^t) \in A(h^t)$, and $a_{-i}^{i*}(h_{-i}^t) \in A_{-i}(h_{-i}^t)$, it follows that for every $u \in U(h^t)$ and every $i \in N$,

$$u_i(a_i^*(h_i^t)) - u_i^{[h_i^t]}(a_i^*(h_i^t)) \geq u_i(a_i) - u_i^{[h_i^t]}(a_i) \text{ for all } a_i \in A_i,$$

and for every $j \in N \setminus \{i\}$,

$$u_j(a_j^{i*}(h_j^t)) - u_j^{[h_j^t]}(a_j^{i*}(h_j^t)) \geq u_j(a_j) - u_j^{[h_j^t]}(a_j) \quad \text{for all} \quad a_j \in A_j .$$

These observations imply that $u^{[h^t]} \in P$ is a universal competitive equilibrium.

Q.E.D.

Parkes and Ungar (2002) and Lahaie and Parkes (2004) showed that there exists a VCG mechanism that is price-based on an ask price procedure if and only if any realized history reveals a universal competitive equilibrium. Theorem 5 and Proposition 7 demonstrate a tractable method for determining whether the history reveals a universal competitive equilibrium and for calculating the universal competitive equilibrium.

The previous works such as Parkes and Ungar (2002), Lahaie and Parkes (2004), and Mishra and Parkes (2007) investigated ask price procedures, according to which, the auctioneer starts with asking the null price vectors, and eventually asks a universal competitive equilibrium. The construction in Section 7 could be regarded as a generalization of these works; this paper does not necessarily require the auctioneer to start with asking the null price vectors nor does it require that the auctioneer ask for a universal competitive equilibrium to stop asking the price at any time.

We can consider an ask price procedure according to which the auctioneer always adjusts his ask prices to the profile of representative valuation functions. In this case, whenever the procedure is successful in achieving VCG outcomes, the auctioneer could obviously ask a universal competitive equilibrium at any time to stop asking the price.

Moreover, according to an ask price procedure, the auctioneer may, at the time of stopping to ask the price, ask a universal competitive equilibrium that is different from the profile of representative valuation functions. In this case, we must note that although the asked universal competitive equilibrium can never be the sufficient statistics for privacy preservation, the resulting profile of representative valuations could be the sufficient statistics for privacy preservation.

10. Auctioneer's Discretion

Let us call any combination of an allocation and a profile of side payments, $(a,s)\in A\times R^n$, an *outcome,* where $s=(s_i)_{i\in N}\in R^n$, and the real number s_i implies a side payment from buyer i to the seller. This section will permit the auctioneer to select an ask price procedure in a discretionary manner with some restrictions. With the discretion, he does not need to make a pre-play agreement with the seller and the buyers in terms of the finer details of the ask price procedure. We will argue that this discretion improves upon the preservation of the buyers' privacy; in particular, the auctioneer can make the selection of the ask price procedure contingent on his *private information*.

An outcome (a,s) is said to be *VCG* for $u\in U$ if

$$a\in A^*(u),$$

and

$$s_i=\max_{a'_{-i}\in A^i_{-i}}\sum_{j\in N\setminus\{i\}}u_j(a'_j)-\sum_{j\in N\setminus\{i\}}u_j(a_j)\quad\text{for all } i\in N.$$

As is evident, a mechanism G is VCG if and only if $(g(u),q(u))$ is VCG for each $u\in U$. Assume that the auctioneer is restricted to the selection of a VCG outcome, and that he is required to enable the verification of whether the selected outcome is VCG only on the basis of the history that occurs.

To be precise, let $\hat{H}\subset H$ denote the set of all histories $h^t\in H$ satisfying properties (7) and (8). For every $h^t\in\hat{H}$, let $\Gamma(h^t)\subset A\times R^n$ denote the set of all outcomes that are VCG irrespective of $u\in U(h^t)$. The auctioneer makes the pre-play agreement with the buyers and the seller that he will continue to ask price vectors until any history h^t that belongs to $\hat{H}$ occurs. Once $h^t\in\hat{H}$ has occurred, the auctioneer selects an outcome (a,s) from the set $\Gamma(h^t)$, where it was assumed that

$$a\in A(h^t)\bigcap A^*(u^{[h^t]}),$$

and for every $i\in N$, there exists $a^i_{-i}\in A_{-i}(h^t_{-i})\bigcap A^{i*}_{-i}(u^{[h^t_{-i}]}_{-i})$ such that

$$s_i=\sum_{j\in N\setminus\{i\}}u_j(a^i_j)-\sum_{j\in N\setminus\{i\}}u_j(a_j).$$

Let Φ denote a subset of connected ask price procedures for each of which there exists a VCG mechanism that is price-based. The selection of an ask price procedure from Φ is within the auctioneer's discretion. Because of this discretion, the auctioneer can utilize his private information to further preserve the buyers' privacy. Let Ω denote the set of *private signals* for the auctioneer. Let $\xi : U \to \Omega$ denote the informational structure concerning the occurrence of this signal. The auctioneer receives a private signal $\omega = \xi(u) \in \Omega$ that is contingent on the profile of valuation functions u. Let $\lambda : \Omega \to \Phi$ describe the manner in which the auctioneer makes the selection of an ask price procedure, $\lambda(\omega) = (\gamma, \lambda, T) \in \Phi$, contingent on his private signal, ω. The auctioneer selects the ask price procedure (γ, λ, T), according to which, he continues to ask prices until the history $h^{T(u)}(u, \gamma, \lambda) \in \hat{H}$ occurs.

Immediately after observing this history, he stops asking price vectors and selects an outcome that is VCG irrespective of $u \in U(h^t)$ according to a function $\beta : \Omega \times \hat{H} \to A \times R^n$, that is, selects $\beta(\xi(u), h^t) \in A \times R^n$, where it was assumed that

$$\beta(\xi(u), h^t) \in \Gamma(h^t).$$

The mechanism $\hat{G} = (\hat{g}, \hat{q})$ that the auctioneer implements could be specified by

$$(\hat{g}(u), \hat{q}(u)) = \beta(\xi(u), h^{T(u)}(u, \gamma, \lambda)) \quad \text{for all} \quad u \in U,$$

where we denoted that $(\gamma, \lambda, T) = \lambda(\xi(u))$. Clearly, this mechanism is VCG.

It is important to note that the selections of the functions λ and β, that is, the signal-contingent selection of ask price procedure and outcomes, are within the auctioneer's discretion. The auctioneer's discretion eliminates the complexity of descriptions such as Ω, ξ, λ, and β from the pre-play agreement. The buyers and the seller do not even need to know Ω or ξ.

For example, consider the situation wherein the auctioneer possesses complete information regarding the profile of valuation functions:

$$\Omega = U, \text{ and } \xi(u) = u \quad \text{for all} \quad u \in U.$$

Assume that the auctioneer can select any connected ask price procedure that has a price-based VCG mechanism, that is, Φ includes all such connected ask price procedures. With this assumption, we can specify a function $\lambda : \Omega \to \Phi$ in a way that for every $u \in U$, the profile of price vectors that the auctioneer asks in the initial time 0,

$\gamma(h^0) \in \prod_{i \in N} P_i$, is a universal competitive equilibrium, where $\lambda(\xi(u)) = \lambda(u) = (\gamma, \lambda, T)$. The auctioneer can immediately identify the revelation of the efficient allocation and the efficient allocation without any buyer. The stopping time $T(u)$ can be selected to be as close to zero as possible. It is clear from the argument in Section 7 that for any profile of price vectors $p \in \prod_{i \in N} P_i$, there exists an ask price procedure (γ, λ, T) in Φ such that $\gamma(h^0) = p$. Since the auctioneer possesses full information regarding the profile of valuation functions, he can select any universal competitive equilibrium as the profile of a price vector to be asked in the initial time 0, evading information leakages.

11. Conclusion

We investigated the combinatorial auction problem from a practical perspective. With the assumption of connectedness, we demonstrated a tractable calculation method for elucidating whether there exists a VCG mechanism that is price-based on an arbitrary given ask price procedure and for explicitly deriving such a VCG mechanism. The concept of representative valuation functions played the central role in this method, which was easily calculated on the basis of the history as the history-consistent minimal relative valuations. All the auctioneer had to do for these elucidations was to examine the representative valuation functions; it was necessary and sufficient that the efficient allocations with and without any single buyer associated with the profile of representative valuation functions were revealed in the history.

The profile of representative valuation functions could be a universal competitive equilibrium. It could be the sufficient statistics for the extent to which the information regarding the buyers' valuation functions were leaked to the public. Any connected ask price procedure, which, without stopping the asking of prices, eventually makes the buyers reveal all packages in the long run, could successfully gather sufficient information for implementing a VCG mechanism. We have also argued that the auctioneer's discretion played a significant role in further preserving the buyers' privacy.

We can expect the concept of representative valuations to play decisive roles, even in various incentive problems concerning compatibility with complexity and privacy preservation other than the problem of price-based VCG mechanisms. For example, the earlier version of this paper (Matsushima (2010)) investigated price-based core-selecting auction mechanisms; by using the concept of representative valuation functions, along with the assumption of connectedness, the paper showed how a universal Nash equilibrium that is compatible with core selection could be implemented.

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