CIRJE-F-803

A Robust Estimation of Realized Volatility and Covariance with Micro-market Adjustments and Round-off Errors

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June 2011

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June 15, 2011

Abstract

For estimating the realized volatility and covariance by using high frequency data, Kunitomo and Sato (2008a,b) have proposed the Separating Information Maximum Likelihood (SIML) method when there are micro-market noises. The SIML estimator has reasonable asymptotic properties; it is consistent and it has the asymptotic normality (or the stable convergence in the general case) when the sample size is large under general conditions with *non-Gaussian processes* or *volatility models*. We show that the SIML estimator has the robustness properties in the sense that it is consistent and has the asymptotic normality when there are micro-market (non-linear) adjustments and the round-off errors on the underlying stochastic processes.

Key Words

Realized Volatility with Micro-Market Noise, High-Frequency Data, Separating Information Maximum Likelihood (SIML), micro-market adjustments, Round-off errors, Robustness.

*KSIII-11-5-2. This is a preliminary memorandum on our research project. We thank Wataru Ohta for comments on the previous version

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1. Introduction

Recently a considerable interest has been paid on the estimation problem of the realized volatility by using high-frequency data of financial price processes in financial econometrics. Since the earlier studies often had ignored the presence of micro-market noises in financial markets and there has been a consensus that the micro-market noises are important in high-frequency financial data, several new statistical estimation methods have been developed. See Bandorff-Nielsen et al. (2008) and Malliavin and Mancino (2009) for recent literatures on the related topics. In this respect Kunitomo and Sato (2008a, b) have proposed a new statistical method called the Separating Information Maximum Likelihood (SIML) estimation for estimating the realized volatility and the realized covariance by using high frequency data with the presence of possible micro-market noises. The SIML estimator has reasonable asymptotic properties; it is consistent and it has the asymptotic normality (or the stable convergence in the more general case) when the sample size is large and the data frequency interval is small under a set of regularity conditions for the *non-Gaussian* underlying processes and *volatility models*. Kunitomo and Sato (2010, 2011) have also shown that the SIML estimator has the robustness properties, that is, it is consistent and asymptotically normal even when the noise terms are autocorrelated and/or there are endogenous correlations between the market-noise terms and the (underlying) efficient market price process. There has been recent finance literature on the importance of these aspects in high frequency financial data including Engle and Sun (2007), for instance.

In this paper we shall further investigate the robustness property of the SIML estimation when we have the micro-market adjustment mechanism and the round-off errors in the process of forming the observed prices. The micro-market models including the price adjustments have been discussed in the framework of *micro-market literature* in financial economics (Hansbrouck (2007), for instance). Among many micro-market models, we first take the (linear) adjustment model proposed by Amihud and Mendelson (1987) as a benchmark case in our investigation¿ Then we shall extend it to the linear and nonlinear price adjustment models and we regard a continuous martingale as the hidden intrinsic value of underlying security. A new feature in this context to financial econometrics is to utilize the nonlinear (discrete) time series models and a possible non-linear model is the Simultaneous Switching Autoregressive (SSAR) model developed by Sato and Kunitomo (1996) and Kunitomo and Sato (1999). Also we shall consider the round-off error model as a non-linear transformation for financial price data. The problem of round-off error models has been recently investigated in statistics (Delattre and Jacod (1997), for instance) and it corresponds to the fact that in actual financial markets we have the tick-size effects (the minimum price change size and the minimum order size) as we shall discuss in Section 3, and we often observe bid-ask spreads on securities in the stock markets, for instance.

In these problems there is a common econometric aspect that the observed price can be different from the underlying intrinsic value of the security and we can interpret this phenomenon as a nonlinear transformation from the intrinsic value to the observed prices. We can represent the present situation as the nonlinear statistical models of an unobservable (continuous-time) state process and the observed (discrete-time) stochastic process with measurement errors. When the effects of measurement errors are present, it seems that the existing statistical methods measuring the realized volatility and covariance have some problems to be fixed in various ways. They could handle the problem of our interest, but often they need some special consideration on the underlying mechanism of price process. On the contrary, we shall show that the SIML estimator is robust in these situations; that is, it is consistent and asymptotically normal as the sample size increases under a reasonable set of assumptions. The asymptotic robustness of the SIML method on the realized volatility and covariance has desirable properties over other estimation methods from a large number of data for the underlying continuous stochastic process with micro-market noise in the multivariate non-Gaussian cases. Because the SIML estimation is a simple method, it can be practically used for analyzing the multivariate (high frequency) financial time series.

In Section 2 we introduce the standard model with micro-market noise and the SIML method. We also discuss the asymptotic properties of the SIML estimator in a general situation. Then in Section 3 we give the asymptotic properties of the SIML estimator when there are micro-market adjustments and the round-off error models. In Section 4 we shall report the finite sample properties of the SIML estimator based on a set of simulations. Finally, in Section 5 some brief remarks will be given. Some mathematical details of the proofs of theorems in Section 3 are given in Appendix A, and tables and figures based on simulations in Section 4 are given in Appendix B.

2. The SIML Estimation and its Asymptotic Properties

2.1 The SIML Method

We summarize the derivation of the separating information maximum likelihood (SIML) estimation. Let y_{ij} be the $i-$th observation of the $j-$th (log-) price at t_i^n for $j = 1, \cdots, p; 0 = t_0^n \leq t_1^n \leq \cdots \leq t_n^n = 1$. We set $\mathbf{y}_i = (y_{i1}, \cdots, y_{ip})^{'}$ be a $p \times 1$ vector and $\mathbf{Y}_n = (\mathbf{y}_i^{'})$ be an $n \times p$ matrix of observations. The underlying (vector-valued) continuous process $\mathbf{X}(t)$ $(0 \leq t \leq 1)$, which is not necessarily the same as the observed (log-)prices at t_i^n $(i = 1, \cdots, n)$ and let $\mathbf{v}_i^{'} = (v_{i1}, \cdots, v_{ip})$ be the vector of the micro-market noise at t_i^n. We assume

$$\mathbf{y}_i = \mathbf{x}_i + \mathbf{v}_i \ , \ \mathbf{x}_i = \mathbf{X}(t_i^n) \ , \tag{2.1}$$

and

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \mathbf{\Sigma}_x^{1/2}(s) d\mathbf{B}_s \ \ (0 \leq t \leq 1), \tag{2.2}$$

where $\mathcal{E}(\mathbf{v}_i) = \mathbf{0}$, $\mathcal{E}(\mathbf{v}_i \mathbf{v}_i^{'}) = \mathbf{\Sigma}_v$, $\mathbf{B}_s$ is a $q \times 1$ $(q \geq 1)$ vector of the standard Brownian motions, $\mathbf{\Sigma}_x^{1/2}(s)$ is a $p \times q$ vector function adapted to the $\sigma-$field $\mathcal{F}(\mathbf{x}_r, \mathbf{B}_r, r \leq s)$, and the instantaneous covariance function is $\mathbf{\Sigma}_x(s) = (\sigma_{ij}^{(x)}(s)) = \mathbf{\Sigma}_x^{1/2}(s) \mathbf{\Sigma}_x^{1/2}(s)$ ($\sigma_{ij}^{(x)}(s)$ is the (i, j)-th element of $\mathbf{\Sigma}_x(s)$). The main statistical problem is to estimate the quadratic variations and co-variations

$$\Sigma_x = (\sigma_{ij}^{(x)}) = \int_0^1 \Sigma_x(s)ds \tag{2.3}$$

of the underlying continuous process $\mathbf{X}(t)$ $(0 \le t \le 1)$ and the covariance $\Sigma_v = (\sigma_{ij}^{(v)})$ of the noise process $\mathbf{v}_i$ $(i = 1, \cdots, n)$. We use the notation that $\sigma_{ij}^{(x)}$ and $\sigma_{ij}^{(v)}$ are the (i,j)-th element of Σ_x and Σ_v, respectively.

In order to derive the estimation method, we consider the standard situation when $\mathbf{x}_t$ $(0 \le t \le 1)$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ are independent with $\Sigma_x(s) = \Sigma_x$ (time-invariant), and $\mathbf{v}_i$ are independently, identically and normally distributed as $N_p(\mathbf{0}, \Sigma_v)$. Then given the initial condition $\mathbf{y}_0$, we have

$$\mathbf{Y}_n \sim N_{n\times p}\left(\mathbf{1}_n \cdot \mathbf{y}_0^{'}, \mathbf{I}_n \otimes \Sigma_v + \mathbf{C}_n\mathbf{C}_n^{'} \otimes h_n\Sigma_x\right) \ , \tag{2.4}$$

where

$$\mathbf{C}_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & 0 \\ 1 & \cdots & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \cdots & 0 \\ 0 & \cdots & -1 & 1 & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}^{-1} , \tag{2.5}$$

$\mathbf{1}_n^{'} = (1, \cdots, 1)$ and $h_n = 1/n$ $(= t_i^n - t_{i-1}^n)$.
We transform $\mathbf{Y}_n$ to $\mathbf{Z}_n$ $(= (\mathbf{z}_k^{'}))$ by

$$\mathbf{Z}_n = h_n^{-1/2}\mathbf{P}_n^{'}\mathbf{C}_n^{-1}\left(\mathbf{Y}_n - \bar{\mathbf{Y}}_0\right) \tag{2.6}$$

where $\bar{\mathbf{Y}}_0 = \mathbf{1}_n \cdot \mathbf{y}_0^{'}$, $\mathbf{P}_n = (p_{jk})$ and for $j, k = 1. \cdots, n$,

$$p_{jk} = \sqrt{\frac{2}{n+\frac{1}{2}}}\cos\left[\frac{2\pi}{2n+1}(j-\frac{1}{2})(k-\frac{1}{2})\right] \ . \tag{2.7}$$

Then the likelihood function for (2.4) under the Gaussian noise distribution can be written as

$$L_n(\boldsymbol{\theta}) = \left(\frac{1}{\sqrt{2\pi}}\right)^{np}\prod_{k=1}^{n}|a_{kn}\Sigma_v + \Sigma_x|^{-1/2}e^{\left\{-\frac{1}{2}\mathbf{z}_k^{'}\left(a_{kn}\Sigma_v + \Sigma_x\right)^{-1}\mathbf{z}_k\right\}} , \tag{2.8}$$

where

(2.9) $$a_{kn} = 4n \sin^2 \left[\frac{\pi}{2} \left(\frac{2k-1}{2n+1} \right) \right] \ (k = 1, \cdots, n) \ .$$

Because the ML estimator of unknown parameters is a rather complicated function of all observations and each a_{kn} terms depend on k as well as n, one way to have a simple solution of the problem is to approximate the likelihood function in some sense. When k (or k_n) is small, a_{kn} is small and then we approximate $(-2) \times \log L_n(\boldsymbol{\theta})$ by

(2.10) $$L_{1n}(\boldsymbol{\theta}) = m \log |\boldsymbol{\Sigma}_x| + \sum_{k=1}^{m} \mathbf{z}_k^{'} \boldsymbol{\Sigma}_x^{-1} \mathbf{z}_k \ .$$

Similarly, we consider the corresponding terms when $a_{n+1-k,n}$ is large and approximate $(-2) \times \log L_n(\boldsymbol{\theta})$ by

(2.11) $$L_{2n}(\boldsymbol{\theta}) = \sum_{k=n+1-l}^{n} \log |a_{kn} \boldsymbol{\Sigma}_v| + \sum_{k=n+1-l}^{n} \mathbf{z}_k^{'} [a_{kn} \boldsymbol{\Sigma}_v]^{-1} \mathbf{z}_k \ .$$

Let m and l be dependent on n and we write m_n and l_n formally. Then we define the SIML estimator of $\hat{\boldsymbol{\Sigma}}_x$ by

(2.12) $$\hat{\boldsymbol{\Sigma}}_x = \frac{1}{m_n} \sum_{k=1}^{m_n} \mathbf{z}_k \mathbf{z}_k^{'}$$

and the SIML estimator of $\hat{\boldsymbol{\Sigma}}_v$ by

(2.13) $$\hat{\boldsymbol{\Sigma}}_v = \frac{1}{l_n} \sum_{k=n+1-l_n}^{n} a_{kn}^{-1} \mathbf{z}_k \mathbf{z}_k^{'} \ .$$

The numbers of terms m_n and l_n we use are dependent on n such that $m_n, l_n \to \infty$ as $n \to \infty$. We impose the order requirement that $m_n = O(n^{\alpha})$ $(0 < \alpha < \frac{1}{2})$ and $l_n = O(n^{\beta})$ $(0 < \beta < 1)$ for $\boldsymbol{\Sigma}_x$ and $\boldsymbol{\Sigma}_v$, respectively.

2.2 Asymptotic Properties of the SIML estimator in the Simple Case

Since the SIML estimator has a simple representation, it is not difficult to derive the asymptotic properties of the SIML estimator. In order to make our arguments clear, we first consider the asymptotic normality of the SIML estimator of the realized volatility and the realized covariance in the simple case. However, it is appropriate here to stress the fact that we do not assume the Gaussainity on the noise process to develop the analysis of the asymptotic properties of the SIML estimator.

Let $\mathbf{r}_i = \mathbf{x}_i - \mathbf{x}_{i-1}$ and the (constant) covariance matrix is given by

$$\mathcal{E}\left[n\ \mathbf{r}_i\mathbf{r}_i^{'}|\mathcal{F}_{n,i-1}\right] = \mathbf{\Sigma}_x \tag{2.14}$$

for all i ($i = 1,\cdots,n$). When $\mathbf{\Sigma}_x^{1/2}(s)$ $(0 \leq s \leq 1)$ does not depend on s, we write $\mathbf{\Sigma}_x^{1/2}(s) = \mathbf{\Sigma}_x^{1/2}$) and the realized covariance matrix $\mathbf{\Sigma}_x = (\sigma_{gh}^{(x)})$ is a constant (non-negative definite) matrix. In the standard model Kunitomo and Sato (2008a) have shown the next result.

Theorem 2.1 : We assume that $\mathbf{x}_i$ and $\mathbf{v}_i$ ($i = 1,\cdots,n$) are independent and they follow (2.1) and (2.2) with $\mathbf{\Sigma}_x(s) = \mathbf{\Sigma}_x$ (positive definite) for $s \in [0,1]$, $\mathcal{E}[||\sqrt{n}\mathbf{r}_i||^4] < \infty$ and $\mathcal{E}[||\mathbf{v}_i||^4] < \infty$. Define the SIML estimator $\hat{\mathbf{\Sigma}}_x = (\hat{\sigma}_{gh}^{(x)})$ of $\mathbf{\Sigma}_x = (\sigma_{gh}^{(x)})$ and $\hat{\mathbf{\Sigma}}_v = (\hat{\sigma}_{gh}^{(v)})$ of $\mathbf{\Sigma}_v = (\sigma_{gh}^{(v)})$ by (2.12) and (2.13), respectively.
(i) For $m_n = n^{\alpha}$ and $0 < \alpha < 0.5$, as $n \longrightarrow \infty$

$$\hat{\mathbf{\Sigma}}_x - \mathbf{\Sigma}_x \xrightarrow{p} \mathbf{O}\ . \tag{2.15}$$

(ii) For $m_n = n^{\alpha}$ and $0 < \alpha < 0.4$, as $n \longrightarrow \infty$

$$\sqrt{m_n}\left[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}\right] \xrightarrow{w} N\left(0, \sigma_{gg}^{(x)}\sigma_{hh}^{(x)} + \left[\sigma_{gh}^{(x)}\right]^2\right)\ . \tag{2.16}$$

The covariance of the limiting distributions of $\sqrt{m_n}[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}]$ and $\sqrt{m_n}[\hat{\sigma}_{kl}^{(x)} - \sigma_{kl}^{(x)}]$ is given by $\sigma_{gk}^{(x)}\sigma_{hl}^{(x)} + \sigma_{gl}^{(x)}\sigma_{hk}^{(x)}$ $(g,h,k,l = 1,\cdots,p)$.

2.3 Asymptotic Properties of the SIML estimator when the Instantaneous Covariance function is Time-varying

It is important to investigate the asymptotic properties of the SIML estimator when the instantaneous volatility function $\mathbf{\Sigma}_x(s)$ is not constant over time. When the realized volatility is a positive (deterministic) constant a.s. (i.e. $\sigma_{gh}^{(x)} = \int_0^1 \sigma_{gh}^{(x)}(s)ds$ is not stochastic) while the instantaneous covariance function is time varying, we have the consistency and the asymptotic normality of the SIML estimator as $n \to \infty$. For the deterministic time varying case, the asymptotic properties of the SIML estimator can be summarized as follows. (The proof has been given in Kunitomo and Sato (2010).)

Theorem 2.2 : We assume that $\mathbf{x}_i$ and $\mathbf{v}_i$ $(i = 1, \cdots, n)$ in (2.1) and (2.2) are independent, $\boldsymbol{\Sigma}_x = \int_0^1 \boldsymbol{\Sigma}_x(s)ds$ is a constant (or deterministic) positive definite matrix, $\mathcal{E}[\|\sqrt{n}\mathbf{r}_i\|^4] < \infty$ and $\mathcal{E}[\|\mathbf{v}_i\|^4] < \infty$. Define the SIML estimator $\hat{\boldsymbol{\Sigma}}_x = (\hat{\sigma}_{gh}^{(x)})$ of $\boldsymbol{\Sigma}_x = (\sigma_{gh}^{(x)})$ by (2.12) and (2.13), respectively.
(i) For $m_n = n^{\alpha}$ and $0 < \alpha < 0.5$, as $n \longrightarrow \infty$

$$\hat{\boldsymbol{\Sigma}}_x - \boldsymbol{\Sigma}_x \xrightarrow{p} \mathbf{O} . \tag{2.17}$$

(ii) For $m_n = n^{\alpha}$ and $0 < \alpha < 0.4$, as $n \longrightarrow \infty$

$$\sqrt{m_n}\left[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}\right] \xrightarrow{d} N\left[0, V_{gh}\right] , \tag{2.18}$$

provided that

$$\begin{aligned} V_{gh,n} &= \left[\int_0^1 \sigma_{gg}^{(x)}(s)ds\right]\left[\int_0^1 \sigma_{hh}^{(x)}(s)ds\right] + \left[\int_0^1 \sigma_{gh}^{(x)}(s)ds\right]^2 \\ &+ \sum_{i,j=1}^{n}(m_n c_{ij}^2 - 1)\left[\int_{t_{i-1}}^{t_i} \sigma_{gg}^{(x)}(s)ds \int_{t_{j-1}}^{t_j} \sigma_{hh}^{(x)}(s)ds + \int_{t_{i-1}}^{t_i} \sigma_{gh}^{(x)}(s)ds \int_{t_{j-1}}^{t_j} \sigma_{gh}^{(x)}(s)ds\right], \end{aligned}$$

converges to V_{gh}, which is positive constant and for $i, j = 1, \cdots, n$,

$$c_{ij} = \frac{1}{m_n}\sum_{k=1}^{m_n}\{\cos[\frac{2\pi}{2n+1}(i+j-1)(k-\frac{1}{2})] + \cos[\frac{2\pi}{2n+1}(i-j)(k-\frac{1}{2})]\} . \tag{2.19}$$

There are some remarks on the limiting distribution of the SIML estimator and its asymptotic covariance formula in Theorem 2.2. The quantity $V_{gh.n}^{(2)}$ defined by

$$V_{gh.n}^{(2)} = \sum_{i,j=1}^{n}(m_n c_{ij}^2 - 1)\left[\int_{t_{i-1}}^{t_i} \sigma_{gg}^{(x)}(s)ds \int_{t_{j-1}}^{t_j} \sigma_{hh}^{(x)}(s)ds + \int_{t_{i-1}}^{t_i} \sigma_{gh}^{(x)}(s)ds \int_{t_{j-1}}^{t_j} \sigma_{gh}^{(x)}(s)ds\right]$$

is bounded because $\int_0^1 \sigma_{gg}^{(x)}(s)ds$ is bounded.

Then it may be reasonable to assume the convergence of $V_{gh.n}^{(2)}$ to the second part of V_{gh} ($V_{gh}^{(2)}$, say). When the instantaneous covariance $\sigma_{gh}^{(x)}(s) = \sigma_{gh}^{(x)}$ is constant, then

$$V_{gh} = \left[\int_0^1 \sigma_{gg}^{(x)}(s)ds\right]\left[\int_0^1 \sigma_{hh}^{(x)}(s)ds\right] + \left[\int_0^1 \sigma_{gh}^{(x)}(s)ds\right]^2 , \tag{2.20}$$

which is equivalent to the asymptotic variance in (2.16). Furthermore, when $p = 1$, we have $\sigma_{gg} = \sigma^2$ and $V_{gg} = 2\sigma_{gg}$.

When $\mathbf{\Sigma}_x$ is a random matrix, we need the concept of stable convergence. The results of Theorem 2.2 can be held in the proper stochastic case with an additional assumption.

Theorem 2.3 : We assume that $\mathbf{x}_i$ and $\mathbf{v}_i$ ($i = 1, \cdots, n$) in (2.1) and (2.2) are independent and $\mathbf{\Sigma}_(s) > 0$ (positive definite). Additionally we assume that each elements of $\mathbf{\Sigma}_x(s)$ ($0 \le s \le 1$) and $\mathbf{\Sigma}_x = \int_0^1 \mathbf{\Sigma}_x(s)ds$ are *bounded* and $\mathcal{E}[\|\mathbf{v}_i\|^4] < \infty$. Define the SIML estimator $\hat{\mathbf{\Sigma}}_x = (\hat{\sigma}_{gh}^{(x)})$ of $\mathbf{\Sigma}_x = (\sigma_{gh}^{(x)})$ by (2.12).

(i) For $m_n = n^{\alpha}$ and $0 < \alpha < 0.5$, as $n \longrightarrow \infty$

$$\hat{\mathbf{\Sigma}}_x - \mathbf{\Sigma}_x \xrightarrow{p} \mathbf{O} . \tag{2.21}$$

(ii) For $m_n = n^{\alpha}$ and $0 < \alpha < 0.4$, as $n \longrightarrow \infty$ we have the weak convergence

$$Z_{gh.n} = \sqrt{m_n}\left[\hat{\sigma}_{gh}^{(x)} - \sigma_{gh}^{(x)}\right] \xrightarrow{w} Z_{gh}^* , \tag{2.22}$$

where the characteristic function $g_n(t) = \mathcal{E}[\exp(itZ_{gh.n})]$ converges to the characteristic function of Z_{gh}^*, which is written as

$$g(t) = \mathcal{E}[e^{-\frac{V_{gh}t^2}{2}}] \tag{2.23}$$

and we assume the probability convergence given by

$$\begin{aligned} V_{gh} &= \left[\int_0^1 \sigma_{gg}^{(x)}(s)ds\right]\left[\int_0^1 \sigma_{hh}^{(x)}(s)ds\right] + \left[\int_0^1 \sigma_{gh}^{(x)}(s)ds\right]^2 \\ &+ \operatorname*{plim}_{n\to\infty} \sum_{i,j=1}^{n} (m_n c_{ij}^2 - 1)\left[\int_{t_{i-1}}^{t_i} \sigma_{gg}^{(x)}(s)ds \int_{t_{j-1}}^{t_j} \sigma_{hh}^{(x)}(s)ds \right. \\ &\qquad \left. + \int_{t_{i-1}}^{t_i} \sigma_{gh}^{(x)}(s)ds \int_{t_{j-1}}^{t_j} \sigma_{gh}^{(x)}(s)ds\right] . \end{aligned} \tag{2.24}$$

3. Robustness under the Micro-market adjustments and the Round-off error models

3.1 A General Formulation

In this section, we shall re-consider the standard model given by (2.1) and (2.2). We set $p = q = 1$ and treat the univariate price process because it may be often rather straightforward to extend the results reported in this section for $p = q = 1$ to the multivariate cases when $p > 1$. One extension of the present problem would be to consider the a sequence of discrete stochastic process given by

$$y(t_i^n) = h\left(x(t), y(t_{i-1}^n), u(t_i^n)\ ,\ 0 \le t \le t_i^n\right)\ , \tag{3.1}$$

where the (unobservable) continuous martingale process $x(t)$ $(0 \le t \le 1)$ is defined by (2.2), $u(t_i^n)$ is the micro-market noise process. For the simplicity we assume that $\mathcal{E}(u(t_i^n)) = 0$, $\mathcal{E}(u(t_i^n)^2) = \sigma_u^2$, and $h(\cdot)$ is a measurable function at $0 = t_0^n \le t_1^n \le \cdots \le t_n^n = 1$ with $t_i^n - t_{i-1}^n = 1/n$ $(i = 1, \cdots, n)$.
There are several special cases of (3.1), which have some interesting aspects for practical applications on modeling the financial markets and the high frequency financial data. As we shall discuss, some of the financial models for micro-market price adjustments and the round-off-errors models for financial prices can be represented as (3.1).

3.2 A Micro-market price Adjustment model

There have been a large number of micro-market models in the area of financial economics in the past which have tried to explain the role of noise traders, insiders, bid-ask spreads, the transaction prices and the associated price adjustment processes. (See Hansbrouck (2007) for the detailed discussions on the major micro-market models in financial economics, for instance.) We illustrate the underlying arguments on the financial markets by showing some figures (Figures 3-1 and 3-2) in Appendix B. For this purpose, we denote that P and Q are the price and the quantity (demand, supply and traded) of a security [1]. When the demand curve and supply surve for a security do not meet as Figure 3.1, there is no transaction occurred at the moment in a financial market. The minimum (desired) supply price level $\bar{P}$ is higher than the maximum (desired) demand price level $\underline{P}$, and then there is a (bid-ask) spread. When there were some information in the supply side indicating that the intrinsic value of a security at t could be less than the latest observed price at $t - \Delta t$ (i.e. $V_t - P_{t-\Delta t} < 0$, $\Delta t > 0$), however, the supply schedule would be shifted down-ward When, however, there were some information in the demand side indicating that the intrinsic value of a seculity at t could be higher than the latest observed price (i.e. $V_t - P_{t-\Delta t} > 0$), the demand schedule would be shifted up-ward. In these situations while the trade of a security occur at the price P^* and the quantity Q^* as in Figure 3.2, the financial market would be under pressure to further price changes.

We set $y_i = P(t_i^n)$ $(i = 1, \cdots, n)$ and $x_i = X(t_i^n)$ $(i = 1, \cdots, n)$ with $p = q = 1$. We consider the (linear) micro-market price adjustment model given by

$$P(t_i^n) - P(t_{i-1}^n) = g\left[X(t_i^n) - P(t_{i-1}^n)\right] + u(t_i^n) \,, \tag{3.2}$$

where $X(t)$ (the intrinsic value of a security at t) and $P(t_i^n)$ (the observed price at t_i^n) are measured in logarithms, the adjustment (constant) coefficient g $(0 < g < 2)$, and $u(t_i^n)$ is an i.i.d. sequence of noise with $\mathcal{E}[u(t_i^n)] = 0$ and $\mathcal{E}[u(t_i^n)^2] = \sigma_u^2$.
We first consider the specific model (3.2), which was originally proposed by Amihud and Mendelson (1987), as an example because it has been one of well-known models involving transaction costs, interactions among market participants and micro-market structure. We shall depart our discussion from the Amihud-Mendelson model because we are mainly interested in the price adjustment dynamics of a security while their main purpose of study was to investigate the micro-market mechanisms by using daily (open-to-open and close-to-close) data. While Amihud and Mendelson (1987) used that $X(t_i^n)$ follows a (discrete) random walk process in the discrete time

[1] This is only an illustration for the exposition, which may be analogous to the current market practice for the periodic call option of the Tokyo Stock Exchange (TSE).

series framework, we assume that $X(t)$ is a (scalar) continuous martingale, which is represented as

$$X(t) = X(t_0^n) + \int_0^t \sigma_s dB_s \ (0 \leq t \leq 1) \ , \tag{3.3}$$

where B_s is the standard Brownian motion on $[0,1]$ and $0 < \int_0^t \sigma_s^2 ds < \infty$ (a.s.).

We consider the situation that we have a sequence of discrete observations $P(t_i^n)$ with $0 = t_0^n < t_1^n < \cdots < t_n^n = 1$ and the main purpose is to estimate the realized volatility of the intrinsic value of the underlying security

$$\Sigma_x = \int_0^1 \sigma_s^2 ds \ . \tag{3.4}$$

We re-express (3.2) as

$$\begin{aligned} P(t_i^n) &= (1-g)P(t_{i-1}^n) + gX(t_i^n) + u(t_i^n) \\ &= g\sum_{j=0}^{i-1}(1-g)^j X(t_{i-j}^n) + \sum_{j=0}^{i-1}(1-g)^j u(t_{i-j}^n) \\ &\qquad + \left[g(1-g)^i X(t_0^n) + (1-g)^i u(t_0^n)\right] \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} (1-g)^j X(t_{i-j}^n) &= (1-g)^j \left[X(t_0^n) + \int_0^{t_{i-j}^n} \sigma_s dB_s\right] \\ &= (1-g)^j X(t_i^n) - (1-g)^j \left[\int_{t_{i-j}^n}^{t_i^n} \sigma_s dB_s\right] \ . \end{aligned} \tag{3.6}$$

Then we have the next result and the proof will be given in Appendix A, which is similar to the one given in Kunitomo and Sato (2010).

Theorem 3.1 : Assume $0 < g < 2$ in (3.2). Define the SIML estimator of the realized volatility of $X(t)$ with $m_n = n^{\alpha}$ $(0 < \alpha < 0.4)$ by (2.12) with $p = 1$. Then the asymptotic distribution of $\sqrt{m_n}\left[\hat{\Sigma}_x - \Sigma_x\right]$ is asymptotically $(m_n, n \to \infty)$ equivalent to the limiting distributions given by Theorem 2.1 Theorem 2.2 and Theorem 2.3 under their assumptions.

We note that the present micro-market (linear) adjustment model is quite similar to the structure of the micro-market model with autocorrelated micro-market noise discussed in Kunitomo and Sato (2010).

3.3 The Round-off-error model

Next, we consider the round-off-errors model with the micro-market noise. One motivation has been the fact that in real financial markets transactions occur with the minimum tick size and the observed price data do not have continuous values. The traded quantity also usually has the minimum size in actual financial markets. For instance, the Nikkei-225-futures, which have been the most important traded derivatives in Japan (as explained in Kunitomo and Sato (2008b)), has the minimum 10 yen size while the Nikkei-225-stock index is around 10,000 yen in the year of 2010. (See Hansbrouck (2007) for the details of major stock markets in the U.S., for instance.) Thus it is quite interesting and important to see the effects of round-off-errors on the estimates of the realized volatility when we have realistic round-off errors. We can illustrate the underlying typical argument on the financial markets by showing Figure 3-2 in Appendix B. When the demand curve and supply curve do meet at a point as Figure 3-2, the quantity Q^* is traded at the price P^*. Still there would be excee demand which could not be traded at the particular moment because of the positive tick-size ($\eta > 0$) and the minimum order size effects, i.e. the number of orders should be integers in actual financial markets.

We assume that

$$P(t_i^n) - P(t_{i-1}^n) = g_\eta \left[X(t_i^n) - P(t_{i-1}^n) + u(t_i^n) \right] \ , \tag{3.7}$$

where $u(t_i^n)$ is an i.i.d. sequence of noise with $\mathcal{E}[u(t_i^n)] = 0, \mathcal{E}[u(t_i^n)^2] = \sigma_u^2$ and the nonlinear function

$$g_\eta(x) = \eta \left[\frac{x}{\eta} \right] \ , \tag{3.8}$$

where $g_\eta(y)$ is the integer part of y and $[y]$ is the largest integer being less than y and η is a small positive constant.

This model corresponds to the micro-market model with the restriction of the minimum price change and η is the parameter of minimum price change. We set $y_i = P(t_i^n)$ and $x_i = X(t_i^n)$ $(i = 1, \cdots, n)$. We represent (3.7) as

$$
\begin{aligned}
(3.9)\quad & P(t_i^n) - X(t_i^n) \\
= \; & g_\eta \left[-(P(t_{i-1}^n) - X(t_{i-1}^n)) + \Delta X(t_i^n) + u(t_i^n) \right] - [P(t_{i-1}^n) - X(t_{i-1}^n) - \Delta X(t_i^n)] \\
= \; & g_\eta^* \left[P(t_{i-1}^n) - X(t_{i-1}^n), \Delta X(t_i^n), u(t_i^n) \right]
\end{aligned}
$$

where

$$\Delta X(t_i^n) = \int_{t_{i-1}^n}^{t_i^n} \sigma_s dB_s \tag{3.10}$$

is a sequence of martingale differences.

Define

$$W(t_i^n) = P(t_i^n) - X(t_i^n) - u(t_i^n) \ . \tag{3.11}$$

If $|P(t_{i-1}^n) - X(t_i^n) - u(t_i^n)| > \eta$, then from (3.7) we have $P(t_i^n) = X(t_i^n) + u(t_i^n)$, which means $W(t_i^n) = 0$. On the other hand, if $|P(t_{i-1}^n) - X(t_i^n) - u(t_i^n)| \le \eta$, then $P(t_i^n) = P(t_{i-1}^n)$ and $|W(t_i^n)| \le \eta$. By defining $v_i = u(t_i^n) + W(t_i^n)$ $(i = 1, \cdots, n)$, we have the condition

$$|W(t_i^n)| \le \eta \ . \tag{3.12}$$

By using the similar arguments to the results reported as Theorems 2.1, 2.2 and 2.3 on the limiting distribution of the realized volatility estimator (Kunitomo and Sato (2010b), we have the next result. (The proof will be given in Appendix A.)

Theorem 3.2 : Assume (3.7), (3.8), and $\eta = \eta_n$ depends on n satisfying

$$\eta_n \sqrt{n} = O(1) \ . \tag{3.13}$$

Define the SIML estimator of the realized volatility of $X(t)$ with $m_n = n^\alpha$ $(0 < \alpha < 0.4)$ by (2.12) with $p = 1$. We write the limiting random variable of the normalized estimator $\sqrt{m_n}\left[\hat{\Sigma}_x - \Sigma_x\right]$ as Σ_η when $n \to \infty$. Then as $\eta \to 0$ the distribution of Σ_η is asymptotically $(m_n, n \to \infty)$ equivalent to the limiting distributions given by Theorem 2.1 Theorem 2.2 and Theorem 2.3 under their assumptions.

We have imposed the condition (3.13) on η, which means that itt is a parameter with small size. This condition could be relaxed because the results of simulations in Section 4 have suggested so. The SIML estimator has the asymptotic robust property against a large number of the round-off-errors models.

3.4 Nonlinear Micro-market price Adjustment models

We shall generalize the linear price adjustment model in Section 3.1 and consider nonlinear price adjustments models. As often discussed in the cases of financial crises in the past several decades, there could be different mechanisms among the up-ward phase of financial prices and the down-ward phase of financial prices. In the context of micro-market models in financial economics, some economists have tried to find econometric models involving transaction costs and micro-market structures. In many stock markets usually there are regulations on the maximum limits of down-ward price movements within a day, for instance. One common approach in financial econometrics has been to build statistical models with asymmetrical movements of instantaneous volatility and covariance. The present approach is slightly different from the standard one because we try to consider the micro-market price adjustment processes directly. As an example of the discrete time series modeling of the nonlinear price adjustment model of the security price, we take a non-linear version of (3.2) with

$$g(x) = g_1 x \mathrm{I}(x \geq 0) + g_2 x \mathrm{I}(x < 0) \ , \tag{3.14}$$

where g_i $(i = 1, 2)$ are some constants and $\mathrm{I}(\cdot)$ is the indicator function. This has been called the SSAR (simultaneous switching autoregressive) model, whic have been investigated by Sato and Kunitomo (1996) and Kunitomo and Sato (1999). It is related to one of the threshold autoregressive models in the non-linear time series analysis. A set of sufficient conditions for the geometric ergodicity of the price process is given by

$$g_1 > 0 \ , \ g_2 > 0 \ , \ (1 - g_1)(1 - g_2) < 1 \ . \tag{3.15}$$

This condition has been discussed by Kunitomo and Sato (1999) in the context of nonlinear time series analysis. If we set $g_1 = g_2 = g$, then we have the linear adjustment case and the geometrically ergodicity condition is given by $0 < g < 2$, which was assumed in Theorem 3.1.

More generally, we consider the model

$$P(t_i^n) - P(t_{i-1}^n) = g\left[X(t_i^n) - P(t_{i-1}^n)\right] + u(t_{i-1}^n)\ , \tag{3.16}$$

where $u(t_i^n)$ is an i.i.d. sequence of noise with $\mathcal{E}[u(t_i^n)] = 0$ and $\mathcal{E}[u(t_i^n)^2] = \sigma_u^2$. We set $y_i = P(t_i^n)$ and $x_i = X(t_i^n)$ by using the notation in Section 2 and define a sequence of martingale differences by

$$\Delta X(t_i^n) = X(t_i^n) - X(t_{i-1}^n) = \int_{t_{i-1}^n}^{t_i^n} \sigma_s dB_s\ . \tag{3.17}$$

From (3.15) and (3.16), let

$$V(t_i^n) = P(t_i^n) - X(t_i^n) - u(t_i^n) \tag{3.18}$$

and $w(t_i^n) = -\Delta X(t_i^n) + u(t_{i-1}^n)$. Then we have

$$\begin{aligned} V(t_i^n) &= V(t_{i-1}^n) + w(t_i^n) + g\left[-V(t_{i-1}^n) - w(t_i^n)\right] \\ &= g^*\left[V(t_{i-1}^n) + w(t_i^n)\right]\ , \end{aligned} \tag{3.19}$$

where $g^*(z) = z + g(-z)$, $\mathcal{E}[w(t_i^n)] = 0$, $\mathcal{E}[w(t_i^n)^2] < \infty$ and $\mathcal{E}[V(t_{i-1}^n)w(t_i^n)] = 0$. Because the discrete time series $V(t_i^n)$ satisfies the stochastic difference equation (3.19), it is a Markovian process. In order to have the desired result, we need a set of sufficient conditions, which are some type of ergodic conditions. We summarize our results under some additional conditions with the nonlinear price adjustments and the proof will be given in Appendix A.

Theorem 3.3 : For the non-linear time series process $V(t_i^n)$ satisfying (3.18) and (3.19), we assume that there exist functions $\rho_1(\,\cdot\,)$ and $\rho_2(\,\cdot,\cdot\,)$ such that

$$\mathrm{Cov}[V(t_i^n), V(t_j^n)] = c_1\rho_1(|i-j|)\ , \tag{3.20}$$

where c_1 is a (positive) constant and $\sum_{s=0}^{\infty} \rho_1(s) < \infty$ and

$$\text{Cov}\left[V(t_i^n)V(t_{i'}^n), V(t_j^n)V(t_{j'}^n))\right] = c_2\rho_2(|i - i^{'}|, |j - j^{'}|) , \tag{3.21}$$

where c_2 is a (positive) constant and $\sum_{s,s'=0}^{\infty} \rho_2(s, s^{'}) < \infty$.
Define the SIML estimator of the realized volatility of $P(t_i^n)$ with $m_n = n^{\alpha}$ $(0 < \alpha < 0.4)$ by (2.12) with $p = 1$. Then the asymptotic distribution of $\sqrt{m_n}\left[\hat{\Sigma}_x - \Sigma_x\right]$ is asymptotically (as $m_n, n \to \infty$) equivalent to the limiting distributions given by Theorem 2.1 Theorem 2.2 and Theorem 2.3 under their assumptions.

In the above theorem we impose a set of sufficient conditions as (3.20) and (3.21), which may be relaxed.

A simple example is the linear case when $g(x) = c\ x$ (c is a constant with $0 < c < 2$ and v_i are weakly dependent process. It is straightforward to have (3.20) and (3.21) in this case. The second example is the SSAR(1) model with (3.15). It seems that we need more stringent conditions than (3.15) to have (3.20) and (3.21). There can be a large number of non-linear models for $X(t_i^n)$ and $P(t_i^n)$, and the sufficient conditions for the desired results have been under further investigation.

4. Simulations

We have investigated the robust properties of the SIML estimator for the realized variance based on a set of simulations and the number of replications is 1000. We have taken 20,000, and we have chosen $\alpha = 0.4$ and $\beta = 0.8$. The details of the simulation procedure are similar to the corresponding ones reported by Kunitomo and Sato (2008a, b).

In our simulation we consider several cases when the observations are the sum of signal and micro-market noise when $p = 1$. The the volatility function ($\mathbf{\Sigma}_x(s) = \sigma_x^2(s)$) is given by

$$\sigma_x^2(s) = \sigma(0)^2\left[a_0 + a_1 s + a_2 s^2\right] , \tag{4.1}$$

where a_i $(i = 0, 1, 2)$ are constants and we have some restrictions such that $\sigma_x(s)^2 > 0$ for $s \in [0, 1]$. It is a typical time varying (but deterministic) case and the realized variance $\mathbf{\Sigma}_x = \sigma_x^2$ is given by

$$\sigma_x^2 = \int_0^1 \sigma_x(s)^2 ds = \sigma_x(0)^2 \left[a_0 + \frac{a_1}{2} + \frac{a_2}{3} \right] . \tag{4.2}$$

In this example we have taken several intra-day volatility patterns including the flat (or constant) volatility, the monotone (decreasing or increasing) movements and the U-shaped movements.

Among many Monte-Carlo simulations, we summarize our main results as Tables of Appendix B. We have used several models in the form of (3.1) and each model corresponds to

Model 1 $\quad h_1(x, y, u) = y + g(x - y) + u \ (g : \text{a constant})$,

Model 2 $\quad h_2(x, y, u) = y + g_\eta(x - y + u) \ \ (g_\eta(\cdot) \text{ is } (3.8))$,

Model 3 $\quad h_3(x, y, u) = y + g_\eta(x - y) + u \ \ (g_\eta(\cdot) \text{ is } (3.8))$,

Model 4 $\quad h_4(x, y, u) = y + u + \begin{cases} g_1(x - y) & \text{if } y \geq 0 \ (g_1 : \text{a constant}) \\ g_2(x - y) & \text{if } y < 0 \ (g_2 : \text{a constant}) \end{cases}$,

Model 5 $\quad h_5(x, y, u) = y + \left[g_1 + g_2 \exp(-\gamma |x - y|^2) \right] (x - y) \ (g_1, g_2 : \text{constants})$,

Model 6 $\quad h_6(x, y, u) = y + g_1 \sin\left(g_2(x - y)\right) \ \ (g_1, g_2 : \text{constants})$,

Model 7 $\quad h_7(x, y, u) = y + h_2 \ o \ h_4 \ o \ h_1(x, y, u)$,

respectively.

Model 1 is the standard model when $g = 1$. When $0 < g < 2$, Model 1 corresponds to the linear model with the micro-market adjustment. Model 2 and Model 3 are the models with the round-off errors. Model 2 is the standard round-off model and Model 3 has a more complicated nonlinearity. Model 4 and Model 5 are the SSAR model and the exponential AR model, which have been known as nonlinear (discrete) time series models. Model 6 is an artificial nonlinear model with a trigonometric function. Model 7 is a combination of three nonlinear models, which corresponds to the most complicated nonlinearity in our examples.

For a comparison we have calculated the historical volatility (HI) estimates and the Realized Kernel (RK) estimates, which were developed by Bandorff-Nielsen et al. (2008). It is because there is a natural question on the comparison of the HI estimator, RK estimator and the SIML estimator, then we can compare three methods in each tables. In order to make a fair comparison we have tried to follow the recommendation by Bandorff-Nielsen et al. (2008) on the choice of kernel (Tukey-Hanning) and the band width parameter H. One important issue in the RK method has been to choose H, which depends on the noise variance and the instantaneous variance and we can interpret as $H = c\sqrt{\sigma_u^2/[\sigma_x^2/n]}$ when $p = 1$. We have found that the RK estimation gives a reasonable estimate if we had taken the reasonable value of the key parameter H. In most cases the bias and the variance of the RK estimator are larger than the corresponding values of the SIML estimator. Overall the estimates of the SIML method are quite stable and robust against the possible values of the variance ratio even in the nonlinear situations we have considered.

For Model-1, the estimates obtained by historical-volatility (H-vol) are badly-biased, which have been known in the analysis of high frequency data. Actually, the values of H-vol are badly-biased in all cases of our simulations. Both the SIML method and the RK method give reasonable estimates and the variance of the RK estimator is sometimes smaller than the SIML estimator. (See Figures B1-B4.) For Model-1, however, the RK estimation sometimes gives biased-estimates while the SIML estimation gives reasonable estimates. (See Figure B5.) For Model-2 and Model-3, the RK estimation often gives biased-estimates while the SIML estimation gives reasonable estimates. (See Figures B6-B8.) Contrary to our conjecture, for Model-4 and Model-5 both the SIML and the RK estimations often give reasonable results. Finally, for Model-6 and Model-7 the RL estimation sometimes give biased estimates while the SIML estimation gives reasonable estimates.

By examining these results of our simulations we can conclude that we can estimate both the realized volatility of the hidden martingale part. It may be surprising to find that the SIML method gives reasonable estimates even when we have non-linear transformations of the original unobservable security (intrinsic) values. We have conducted a number of further simulations, but the results are quite similar as we have reported in this section.

5. Conclusions

In this paper, we have shown that the Separating Information Maximum Likelihood (SIML) estimator has the asymptotic robustness in the sense that it is consistent and it has the asymptotic normality under a fairly general conditions even when the standard conditions are not satisfied. They include not only the cases when the micro-market noises are possibly autocorrelated and they are endogenously correlated with the underlying continuous signal process, but also the cases when the micro-market structure has the nonlinear adjustments and the round-off errors under a set of reasonable assumptions. The micro-market factors in actual financial markets are common in the sense that we have the minimum price change and the minimum order size rules; we often observe the bid-ask differences in stock markets, for instance. Therefore the robustness of the estimation methods of the realized volatility and covariance has been quite important. By conducting large number of simulations, we have confirmed that the SIML estimator has reasonable robust properties in finite samples even in these non-standard situations.

As a concluding remark, we should stress on the fact that the SIML estimator is very simple and it can be practically used not only for the realized volatility but also the realized covariance and the hedging coefficients from the multivariate high frequency financial series. Some applications on the analysis of stock-index futures market have been reported in Kunitomo and Sato (2008b, 2011) as illustrations.

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APPENDIX A : Mathematical Derivations of Theorems

In Appendix A, we give some details of the proofs of Theorem 3.1, Theorem 3.2 and Theorem 3.3 in Section 3. Since Theorem 3.3 essentially contains Theorem 3.1, we shall give the proof of Theorem 3.3. (The only difference is the effects of additional terms which are smaller order than $O_p(1)$.)

The proof of Theorem 3.3

(Part-I) We shall investigate the asymptotic properties of the SIML estimator in two steps. The first step is to investigate the conditions that the measurment errors are stochastically negligible.

Consider the case of $p = q = 1$ and define $v_i = V(t_i^n)$ $(i = 1, \cdots, n)$ by (3.18). Then we can represent $y_i = x_i + v_i$, where $y_i = P(t_i^n), x_i = X(t_i^n)$ and $v_i = V(t_i^n)$. We set $u(t_i^n) = 0$ in (3.18) and $\Sigma_x^{1/2}(s) = \sigma_s$ for the resulting simplicity. We write the returns in $(t_{i-1}, t_i]$ as

$$r_i = x_i - x_{i-1} = \int_{t_{i-1}}^{t_i} \sigma_s dB_s \ \ (i = 1, \cdots, n) \tag{A.1}$$

with $0 = t_0 \leq t_1 < \cdots < t_n = 1$ and $t_i - t_{i-1} = 1/n$ $(i = 1, \cdots, n)$. We note that the (instantaneous) volatility function σ_s^2 $(0 \leq s \leq 1)$ and the realized volatility $\Sigma_x = \int_0^1 \sigma_s^2 ds$ can be stochastic.

Let $z_{in}^{(1)}$ and $z_{in}^{(2)}$ $(i = 1, \cdots, n)$ be the i-th elements of

$$\mathbf{z}_n^{(1)} = h_n^{-1/2} \mathbf{P}_n' \mathbf{C}_n^{-1} (\mathbf{x}_n - \bar{\mathbf{y}}_0) \ , \ \mathbf{z}_n^{(2)} = h_n^{-1/2} \mathbf{P}_n' \mathbf{C}_n^{-1} \mathbf{v}_n, \tag{A.2}$$

respectively, where $\mathbf{x}_n = (x_i)$, $\mathbf{v}_n = (v_i)$ and $\mathbf{z}_n = (z_{in})$ are $n \times 1$ vectors with $z_{in} = z_{in}^{(1)} + z_{in}^{(2)}$.

Then by following Kunitomo and Sato (2010), we shall use the arguments developed for investigating the effects of the (possibly) autocorrelated noise term on the asymptotic distribution of $\hat{\Sigma}_x - \Sigma_x$ and $\Sigma_x = \int_0^1 \sigma_s^2 ds$. We shall use the decomposition

$$\text{(A.3)} \quad \sqrt{m_n} \left[\hat{\Sigma}_x - \Sigma_x \right] \ = \ \sqrt{m_n} \left[\frac{1}{m_n} \sum_{k=1}^{m_n} z_{kn}^2 - \Sigma_x \right]$$

$$= \sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n} z_{kn}^{(1)2} - \Sigma_x\right] + \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\mathcal{E}[z_{kn}^{(2)2}]$$
$$+\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\left[z_{kn}^{(2)2} - \mathcal{E}[z_{kn}^{(2)2}]\right] + 2\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\left[z_{kn}^{(1)}z_{kn}^{(2)}\right] .$$

Then we shall investigate the conditions that three terms except the first one of (A.3) are $o_p(1)$. It is because we could estimate the realized volatility consistently as if there were no noise terms in this situation.

Let $\mathbf{b}_k = \mathbf{e}_k^{'}\mathbf{P}_n^{'}\mathbf{C}_n^{-1} = (b_{kj})$ and $\mathbf{e}_k^{'} = (0,\cdots,1,0,\cdots)$ be an $n \times 1$ vector. We write $z_{kn}^{(2)} = \sum_{j=1}^{n} b_{kj}v_{fj}$ and notice that $\sum_{j=1}^{n} b_{kj}b_{k'j} = \delta(k,k')a_{kn}$. Also we shall use the notation that K_i $(i \geq 1)$ are some positive constants.
First by using the condition (3.20) and the Cauchy-Schwartz inequality, we have

$$\text{(A.4)} \qquad \begin{aligned} \mathcal{E}[z_{kn}^{(2)}]^2 &= \mathcal{E}[\sum_{i=1}^{n} b_{ki}v_i \sum_{j=1}^{n} b_{kj}v_j] \\ &\leq \sum_{s=0}^{n} c_1\rho_1(s)[\sum_{i=1}^{n} b_{ki}b_{k,i-l}] \\ &\leq K_1 \times a_{kn} , \end{aligned}$$

provided that $\mathcal{E}(v_i^2)$ are bounded and we use the notation $b_{kj} = 0$ $(j \leq 0)$. By using (2.9) and the relation $\sin x = x - (1/6)x^3 + (1/120)x^5 + o(x^7)$,

$$\text{(A.5)} \qquad \begin{aligned} \frac{1}{m_n}\sum_{k=1}^{m_n} a_{kn} &= \frac{1}{m_n}2n\sum_{k=1}^{m_n}\left[1 - \cos(\pi\frac{2k-1}{2n+1})\right] \\ &= \frac{n}{m_n}\left[2m_n - \frac{\sin\pi\frac{2m_n}{2n+1}}{\sin\pi\frac{1}{2n+1}}\right] \\ &\sim \frac{n}{m_n}\left[2m_n - \frac{(\pi\frac{2m_n}{2n+1}) - \frac{1}{6}(\pi\frac{2m_n}{2n+1})^3}{(\frac{\pi}{2n+1}) - \frac{1}{6}(\frac{\pi}{2n+1})^3}\right] \\ &= O(\frac{m_n^2}{n}) \end{aligned}$$

Then the second term of (A.3) becomes

$$\text{(A.6)} \qquad \frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n}\mathcal{E}[z_{kn}^{(2)}]^2 \leq K_1\frac{1}{\sqrt{m_n}}\sum_{k=1}^{m_n} a_{kn} = O(\frac{m_n^{5/2}}{n})$$

if $0 < \alpha < 0.4$.

For the fourth term of (A.3),

$$
\begin{aligned}
\mathcal{E}\left[\frac{1}{\sqrt{m_n}}\sum_{j=1}^{m_n} z_{kn}^{(1)} z_{kn}^{(2)}\right]^2 &= \frac{1}{m_n}\sum_{k,k'=1}^{m_n}\mathcal{E}\left[z_{kn}^{(1)}z_{k',n}^{(1)}z_{kn}^{(2)}z_{k',n}^{(2)}\right] \\
\text{(A.7)} \qquad &= \frac{1}{m}\sum_{k,k'=1}^{m}\mathcal{E}\left[2\sum_{j,j'=1}^{n} s_{jk}s_{j'k'}\mathcal{E}(r_j r_{j'}|\mathcal{F}_{\min(j,j')})z_{kn}^{(2)}z_{k'n}^{(2)}\right] \\
&= \frac{1}{m_n}\sum_{k,k'=1}^{m_n}\mathcal{E}\left[2\sum_{j=1}^{n} s_{jk}s_{j,k'}\mathcal{E}(r_j^2|\mathcal{F}_{j-1})z_{kn}^{(2)}z_{k',n}^{(2)}\right] \\
&\leq K_2\mathcal{E}\left[(\sup_{0\leq s\leq 1}\sigma_s^2)\frac{2}{n}(\frac{n}{2}+\frac{1}{4})\right]\frac{1}{m_n}\sum_{k,k'=1}^{m_n}\sqrt{a_{kn}}\sqrt{a_{k',n}} \\
&\leq O_p(\sum_{k=1}^{m_n} a_{kn}) \\
&= O_p(\frac{m_n^3}{n}) \ .
\end{aligned}
$$

In the above evaluation we have used the relation

$$
\int_{t_{i-1}}^{t_i}\sigma_s^2 ds \leq \frac{1}{n}\left[\sup_{0\leq s\leq 1}\sigma_s^2\right]
$$

and

$$
|\sum_{j=1}^{n} s_{jk}s_{j,k'}| \leq [\sum_{j=1}^{n} s_{jk}^2] = n/2 + 1/4 \ \text{ for any } k \geq 1 \ .
$$

Hence we need the condition $0 < \alpha < 1/3$. When $\sigma_s = \sigma_x$, i.e., the instantaneous volatility function is constant, (A.7) becomes $O(m_n^2/n)$, which is satisfied if $0 < \alpha < 0.4$.

For the third term of (A.3), we need to consider the variance of

$$
z_{kn}^{(2)2} - \mathcal{E}[z_{kn}^{(2)2}] = \sum_{j,j'=1}^{n} b_{kj}b_{k,j'}\left[v_j v_{,j'} - \mathcal{E}(v_j v_{j'})\right]
$$

and we need to evaluate the expectation of $\left[z_{kn}^{(2)2} - \mathcal{E}[z_{kn}^{(2)2}]\right]\left[z_{k',n}^{(2)2} - \mathcal{E}[z_{k',n}^{(2)2}]\right]$. By using (3.21) and we utilize the fact that

$$
\text{(A.8)} \qquad \sum_{i,i'=1}^{n}\sum_{j,j'=1}^{n} b_{ki}b_{k,i'}b_{k',j}b_{k',j'}\rho_2(|i-i'|,|j-j'|) \sim K_3 \times a_{kn}a_{k',n} \ .
$$

Then by collecting each terms, we obtain

$$\text{(A.9)} \qquad \mathcal{E}\left[\frac{1}{\sqrt{m_n}}\sum_{j=1}^{m_n}(z_{kn}^{(2)2}-\mathcal{E}[z_{kn}^{(2)2}])\right]^2 \le \frac{1}{m_n}\sum_{k,k'=1}^{m_n} a_{kn}a_{k'n} = O(\frac{1}{m_n}\times(\frac{m_n^3}{n})^2) = O(\frac{m_n^5}{n^2})$$

since $\sum_{k=1}^{m} a_{kn} = O(m_n^3/n)$.

Thus the third term of (A.2) is negligible if $0 < \alpha < 0.4$.

(Part-II) The remaining task is to prove the asymptotic normality of the first term of (A.3), that is,

$$\text{(A.10)} \qquad \sqrt{m_n}\left[\frac{1}{m_n}\sum_{k=1}^{m_n} z_{kn}^{(1)2} - \Sigma_x\right]$$

because it is of the order $O_p(1)$. The proof of the asymptotic normality of (A.10) is lengthy, but quite similar to the one given in Kunitomo and Sato (2010) and thus it is omitted here. This completes the proof of Theorem 3.3. **Q.E.D.**

The proof of Theorem 3.2 : The most parts of the proof are very similar to the corresponding ones in the proof of Theorem 3.3. We write $y_i = x_i + v_i, v_i = u_i + w_i$ $(i = 1, \cdots, n)$, where $|w_i| \le \eta_n$. Then we need to check that the effects of a sequence of random variables w_i $(i = 1, \cdots, n)$ are negligible under the additional assumption (3.13) on the threshold parameter η_n (> 0).

We shall illustrate the underlying arguments. From (A.3) and (A.4), we notice that

$$\text{(A.11)} \qquad \left[z_{kn}^{(2)}\right]^2 = \left[\sum_{i=1}^{n} b_{ki}(u_i + w_i)\right]^2 = \left[\sum_{i=1}^{n} b_{ki}u_i\right]^2 + 2\left[\sum_{i=1}^{n} b_{ki}u_i\right]\left[\sum_{i=1}^{n} b_{ki}w_i\right] + \left[\sum_{i=1}^{n} b_{ki}w_i\right]^2 .$$

By using the Cauchy-Swartz inequality, under (3.13) we have

$$\text{(A.12)} \qquad \left[\sum_{i=1}^{n} b_{ki}w_i\right]^2 \le n\eta_n^2 a_{kn} .$$

Then we can find a positive constant such that

$$\text{(A.13)} \qquad \mathcal{E}\left[z_{kn}^{(2)}\right]^2 = \left[\sum_{i=1}^{n} b_{ki}(u_i + w_i)\right]^2 \leq K_4 a_{kn}\left[1 + \eta_n\sqrt{n}\right]^2 \ .$$

By using the similar arguments to other terms in the decomposition of (A.3) as (A.11), we can apply the same arguments as the proof of Theorem 3.3. Then we have the desired result in Theorem 3.2. **Q.E.D.**

APPENDIX B : TABLES and FIGURES

In Tables the variances (σ_x^2) are calculated by the SIML estimation method while H-vol and RK are calculated by the historical volatility estimation and the realized kernel estimation methods, respectively. The true-val means the true parameter value in simulations and mean, SD and MSE correspond to the sample mean, the sample standard deviation and the sample mean squared error of each estimator, respectively.

B-1 : Estimation of Realized Volatility (Model-1)
$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\mathrm{E}-04, g = 0.2)$

n=20000	σ_x^2	H-vol	RK
true-val	1.00E+00	1.00E+00	1.00E+00
mean	1.01E+00	2.33E+00	1.04E+00
SD	1.97E-01	2.32E-02	6.58E-02
MSE	3.89E-02	1.78E+00	6.00E-03

B-2 : Estimation of Realized Volatility (Model-1)
$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\mathrm{E}+00, g = 0.2)$

n=20000	σ_x^2	H-vol	RK
true-val	1.00E+00	1.00E+00	1.00E+00
mean	9.96E-01	1.11E-01	9.71E-01
SD	1.93E-01	2.35E-03	6.30E-02
MSE	3.74E-02	7.90E-01	4.80E-03

B-3 : Estimation of Realized Volatility (Model-1)
$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\mathrm{E}+00, g = 1.5)$

n=20000	σ_x^2	H-vol	RK
true-val	1.00E+00	1.00E+00	1.00E+00
mean	1.00E+00	3.00E+00	1.01E+00
SD	1.94E-01	4.03E-02	6.55E-02
MSE	3.78E-02	4.00E+00	4.34E-03

B-4 : Estimation of Realized Volatility (Model-1)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\text{E} - 05, g = 1.0)$

n=20000	σ_x^2	H-vol	RK
true-val	1.00E+00	1.00E+00	1.00E+00
mean	9.88E-01	1.40E+00	9.97E-01
SD	1.99E-01	1.40E-02	6.53E-02
MSE	3.97E-02	1.60E-01	4.27E-03

B-5 : Estimation of Realized Volatility (Model-1)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\text{E} - 06, g = 0.01)$

n=20000	σ_x^2	H-vol	RK
true-val	1.00E+00	1.00E+00	1.00E+00
mean	8.40E-01	2.51E-02	2.48E-01
SD	1.66E-01	5.41E-04	2.76E-02
MSE	5.31E-02	9.50E-01	5.66E-01

B-6 : Estimation of Realized Volatility (Model-2)

$(a_0 = 7, a_1 = -12, a_2 = 6; \sigma_u^2 = 2.00\text{E} - 02, \eta = 0.5)$

n=20000	σ_x^2	H-vol	RK
true-val	4.50E+01	4.50E+01	4.50E+01
mean	4.60E+01	1.37E+02	5.36E+01
SD	1.05E+01	6.19E+00	3.65E+00
MSE	1.11E+02	8.46E+03	8.68E+01

B-7 : Estimation of Realized Volatility (Model-3)

$(a_0 = 7, a_1 = -12, a_2 = 6; \sigma_u^2 = 1.00\text{E} - 02, \eta = 0.5)$

n=20000	σ_x^2	H-vol	RK
true-val	4.50E+01	4.50E+01	4.50E+01
mean	4.54E+01	3.95E+02	6.19E+01
SD	1.05E+01	6.69E+00	4.07E+00
MSE	1.10E+02	1.22E+05	3.02E+02

B-8 : Estimation of Realized Volatility (Model-3)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\text{E} + 00, \eta = 0.005)$

n=20000	σ_x^2	H-vol	RK
true-val	1.00E+00	1.00E+00	1.00E+00
mean	1.00E+00	6.85E-01	9.97E-01
SD	1.94E-01	8.66E-03	6.21E-02
MSE	3.77E-02	9.92E-02	3.87E-03

B-9 : Estimation of Realized Volatility (Model-4)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\text{E} + 00, g_1 = 0.2, g_2 = 5)$

n=20000	σ_x^2	H-vol	RK
true-val	1.00E+00	1.00E+00	1.00E+00
mean	1.01E+00	2.22E+00	1.01E+00
SD	1.93E-01	6.46E-02	6.25E-02
MSE	3.71E-02	1.49E+00	3.93E-03

B-10 : Estimation of Realized Volatility (Model-4)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\text{E} - 03, g_1 = 0.2, g_2 = 5)$

n=20000	σ_x^2	H-vol	RK
true-val	1.00E+00	1.00E+00	1.00E+00
mean	1.02E+00	6.65E+01	1.11E+00
SD	1.94E-01	1.66E+00	7.46E-02
MSE	3.79E-02	4.30E+03	1.85E-02

B-11 : Estimation of Realized Volatility (Model-5)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\text{E} + 00, g_1 = 1.9, g_2 = -1.7, \gamma = 10000)$

n=20000	σ_x^2	H-vol	RK
true-val	1.00E+00	1.00E+00	1.00E+00
mean	9.99E-01	6.39E+00	1.00E+00
SD	1.92E-01	3.66E-01	6.53E-02
MSE	3.68E-02	2.91E+01	4.26E-03

B-12 : Estimation of Realized Volatility (Model-6)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\mathrm{E}+00, \sin(z * 0.1))$

n=20000	σ_x^2	H-vol	RK
true-val	1.00E+00	1.00E+00	1.00E+00
mean	1.00E+00	5.26E-02	8.32E-01
SD	2.14E-01	2.23E-03	6.79E-02
MSE	4.59E-02	8.97E-01	3.27E-02

B-13 : Estimation of Realized Volatility (Model-6)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\mathrm{E}+00, 0.01 * \sin(z * 100))$

n=20000	σ_x^2	H-vol	RK
true-val	1.00E+00	1.00E+00	1.00E+00
mean	7.67E-01	4.49E-01	7.75E-01
SD	1.79E-01	3.78E-03	6.05E-02
MSE	8.64E-02	3.03E-01	5.41E-02

B-14 : Estimation of Realized Volatility (Model-7)

$(a_0 = 1, a_1 = 0, a_2 = 0; \sigma_u^2 = 1.00\mathrm{E}-04, g_1 = 0.2, g_2 = 5; g = 0.01; \eta = 0.01)$

n=20000	σ_x^2	H-vol	RK
true-val	1.00E+00	1.00E+00	1.00E+00
mean	1.18E+00	3.62E+00	1.81E+00
SD	2.30E-01	1.04E-01	1.16E-01
MSE	8.36E-02	6.85E+00	6.69E-01

In Figures 3.1 and 3.2 P and Q stand for the price and the quantity, respectively. D and S are the demand curve and supply curve, respectively. η in Table 3.2 denotes the minimum tick size and Q^* is the quantity traded in Figure 3.2.

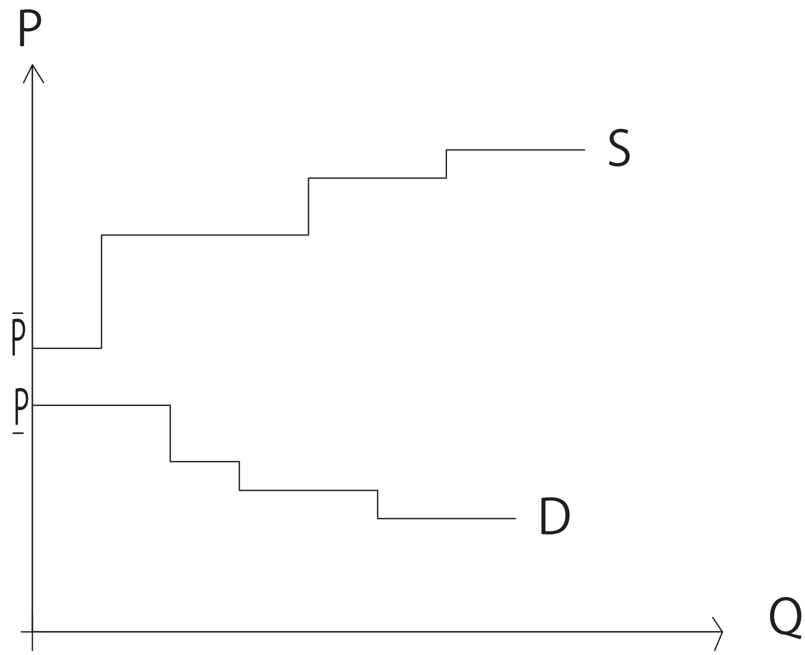

Fig. 3-1

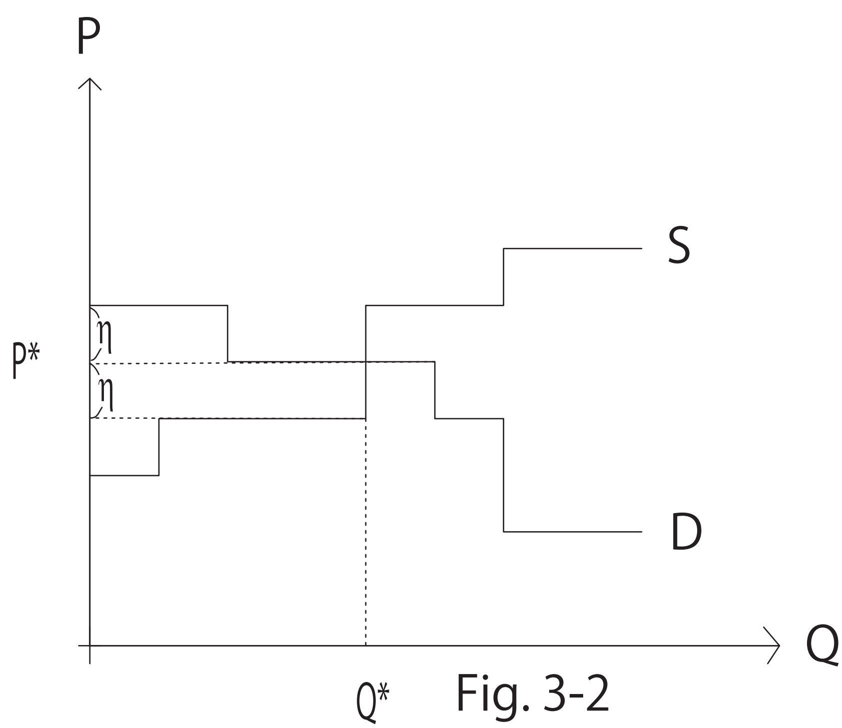

Fig. 3-2